COUNTING PRIME ORBITS IN SHRINKING INTERVALS FOR EXPANDING THURSTON MAPS

ZHIQIANG LI AND XIANGHUI SHI

ABSTRACT. We establish a local central limit theorem for primitive periodic orbits of expanding Thurston maps, providing a fine-scale refinement of the Prime Orbit Theorem in the context of non-uniformly expanding dynamics. Specifically, we count the number of primitive periodic orbits whose Birkhoff sums for a given potential lie within a family of shrinking intervals. For eventually positive, real-valued Hölder continuous potentials that satisfy the strong non-integrability condition, we derive precise asymptotic estimates. In particular, our results apply to postcritically-finite rational maps whose Julia set is the whole Riemann sphere.

CONTENTS

1. INTRODUCTION

Periodic orbits serve as the skeleton of chaotic dynamics, encoding essential information about the system's long-term behavior. A fundamental objective in this field is counting these orbits, a problem analogous to the Prime Number Theorem in number theory. Recently, the dynamical

---

Date: November 28, 2025.

2020 *Mathematics Subject Classification.* Primary: 37D20; Secondary: 37C25, 37C35, 37D35, 37F15, 57M12.

Key words and phrases. expanding Thurston map, postcritically-finite rational map, prime orbits, thermodynamic formalism, Ruelle operator, shrinking targets.

Li and Shi were partially supported by Beijing Natural Science Foundation (JQ25001 and 1214021) and National Natural Science Foundation of China (12471083, 12101017, 12090010, and 12090015).

counterpart—the Prime Orbit Theorem—was successfully established for expanding Thurston maps [LZ24a, LZ24c], which serve as topological models for postcritically-finite rational maps. While this answers the question of "how many" orbits exist, it leaves open the more subtle question of "how they are distributed" with respect to statistical observables. Understanding this fine-scale distribution is crucial for characterizing the fluctuations inherent in the system.

In this paper, we investigate the asymptotic distribution of primitive periodic orbits restricted to shrinking intervals for expanding Thurston maps. We count orbits whose Birkhoff sums for a given potential lie within a prescribed family of shrinking intervals. we obtain a precise asymptotic formula for the number of these constrained orbits, which resemble a local central limit theorem. Unlike the central limit theorem, which describes the distribution of Birkhoff sums on the scale of $\sqrt{n}$, the local central limit theorem can probe the density at a given point. Our main result refines the coarse counting of the Prime Orbit Theorem [LZ24a] for expanding Thurston maps.

The problem of counting orbits in shrinking intervals has been successfully addressed for uniformly expanding systems. For instance, Petkov and Stoyanov [PS12] investigated the distribution of closed orbits for hyperbolic flows, and Sharp and Stylianou [SS22] studied the multipliers and holonomies for hyperbolic rational maps. However, these results rely heavily on the hyperbolicity and smoothness of the underlying systems. The context of non-uniformly expanding dynamics, particularly for branched covering maps, remains largely unexplored. To the best of our knowledge, our work is the first to address this problem in such a setting.

1.1. **Main results.** Let $f\colon S^2 \to S^2$ be an expanding Thurston map and $\phi\colon S^2 \to \mathbb{R}$ be a real-valued Hölder continuous function. The topological 2-sphere S^2 is equipped with a *visual metric* d (see Section 2.3 for details). A periodic orbit $\tau = \bigl\{x, f(x), \dots, f^{n-1}(x)\bigr\}$ (where $f^n(x) = x$) is called *primitive* if $f^m(x) \neq x$ for each integer m with $1 \leqslant m < n$. We denote the set of primitive periodic orbits of f by $\mathfrak{P}(f)$. For each $\tau \in \mathfrak{P}(f)$, we write $\phi(\tau) \coloneqq \sum_{x\in\tau} \phi(x)$.

In this article, we investigate the asymptotic distribution of primitive periodic orbits subject to constraints on their Birkhoff sums. Specifically, for a given number $\alpha \in \mathbb{R}$ and a sequence $\{I_n\}_{n\in\mathbb{N}}$ of intervals contained in a compact set $K \subseteq \mathbb{R}$, we study the asymptotic behavior of

$$\pi_{f,\phi}(n;\alpha, I_n) \coloneqq \operatorname{card}\bigl\{\tau \in \mathfrak{P}_n(f) : \phi(\tau) - n\alpha \in I_n\bigr\}$$

as $n \to +\infty$, where $\mathfrak{P}_n(f) \coloneqq \bigl\{\tau \in \mathfrak{P}(f) : |\tau| = n\bigr\}$.

To obtain precise estimates, we impose specific conditions on the potential ϕ and the sequence $\{I_n\}_{n\in\mathbb{N}}$. We assume that ϕ is *eventually positive* and satisfies the *strong non-integrability condition* (see Definitions 2.11 and 2.13). Furthermore, denoting the length of I_n by $|I_n|$, we assume that the sequence $\bigl\{|I_n|^{-1}\bigr\}_{n\in\mathbb{N}}$ exhibits sub-exponential growth, i.e., $\limsup_{n\to+\infty} \frac{1}{n}\log\bigl(|I_n|^{-1}\bigr) = 0$.

We write $A(n) \sim B(n)$ as $n \to +\infty$ if $\lim_{n\to+\infty} A(n)/B(n) = 1$.

Theorem 1.1. *Let $f\colon S^2 \to S^2$ be an expanding Thurston map and d be a visual metric on S^2 for f. Let $\beta \in (0,1]$ and $\phi \in C^{0,\beta}(S^2,d)$ be an eventually positive real-valued Hölder continuous function satisfying the β-strong non-integrability condition (with respect to f and d). Then there exists a unique positive number $s_0 > 0$ with topological pressure $P(f,-s_0\phi) = 0$ and there exists $N_f \in \mathbb{N}$ depending only on f such that for each $N \in \mathbb{N}$ with $N \geqslant N_f$, the following statement holds for the iterate $F \coloneqq f^N$ and the potential $\Phi \coloneqq \sum_{i=0}^{N-1} \phi \circ f^i$:*

Denote $\alpha \coloneqq \frac{\mathrm{d}}{\mathrm{d}t} P(F,t\Phi)|_{t=-s_0}$ and $\sigma \coloneqq \sqrt{\frac{\mathrm{d}^2}{\mathrm{d}t^2} P(F,t\Phi)|_{t=-s_0}}$. Let $\{I_n\}_{n\in\mathbb{N}}$ be a sequence of intervals contained in a compact set $K \subseteq \mathbb{R}$ with $\bigl\{|I_n|^{-1}\bigr\}_{n\in\mathbb{N}}$ having sub-exponential growth. Then

$$\pi_{F,\Phi}(n;\alpha, I_n) \sim \frac{\int_{I_n} e^{s_0 t}\,\mathrm{d}t}{\sqrt{2\pi}\,\sigma}\,\frac{e^{s_0\alpha n}}{n^{3/2}} \qquad \text{as } n \to +\infty.$$

Recall that a postcritically-finite rational map is expanding if and only if it has no periodic critical points (see [BM17, Proposition 2.3]). Therefore, when we restrict our attention to rational maps, we obtain the following corollary of Theorem 1.1 and Remark 2.3.

Corollary 1.2. *Let $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a postcritically-finite rational map without periodic critical points. Let σ be the chordal metric or the spherical metric on the Riemann sphere $\widehat{\mathbb{C}}$, and $\phi \in C^{0,\beta}\big(\widehat{\mathbb{C}}, \sigma\big)$ be an eventually positive real-valued Hölder continuous function with exponent $\beta \in (0,1]$ satisfying the β-strong non-integrability condition (with respect to f and a visual metric). Then there exists a unique positive number $s_0 > 0$ with topological pressure $P(f, -s_0\phi) = 0$ and there exists $N_f \in \mathbb{N}$ depending only on f such that for each $N \in \mathbb{N}$ with $N \geqslant N_f$, the following statements hold for $F := f^N$ and $\Phi := \sum_{i=0}^{N-1} \phi \circ f^i$:*

Denote $\alpha := \frac{\mathrm{d}}{\mathrm{d}t} P(F, t\Phi)|_{t=-s_0}$ and $\sigma := \sqrt{\frac{\mathrm{d}^2}{\mathrm{d}t^2} P(F, t\Phi)|_{t=-s_0}}$. Let $\{I_n\}_{n\in\mathbb{N}}$ be a sequence of intervals contained in a compact set $K \subseteq \mathbb{R}$ with $\big\{|I_n|^{-1}\big\}_{n\in\mathbb{N}}$ having sub-exponential growth. Then

$$\pi_{F,\Phi}(n; \alpha, I_n) \sim \frac{\int_{I_n} e^{s_0 t}\,\mathrm{d}t}{\sqrt{2\pi}\,\sigma} \frac{e^{s_0 \alpha n}}{n^{3/2}} \qquad \text{as } n \to +\infty.$$

1.2. **Strategy and organization.** Our approach relies on a combination of thermodynamic formalism and operator theory, specifically adapted to the branched covering setting.

The main technical obstacle in studying Thurston maps is the presence of critical points, which disrupts the functional analytic properties of the standard Ruelle transfer operator. To overcome this, we employ the *split Ruelle operators* introduced in [LZ24c]. The idea is to decompose the sphere into "black" and "white" tiles (based on a checkerboard coloring induced by an invariant Jordan curve) and define a pair of operators acting on functions supported on these tiles. This construction effectively "unfolds" the singularities, allowing us to recover good spectral properties.

To obtain the precise asymptotics required for the local central limit theorem, we need to control the decay of the characteristic function of the Birkhoff sums. In terms of operator theory, this translates to bounding the spectral radius of the twisted transfer operator $\mathbb{L}_{s\phi}$ as the complex parameter s moves along the imaginary axis. The detailed estimates are separated into three parts: the unbounded part, the bounded part, and the local part. For the unbounded part, we employ Dolgopyat-type estimates for the split Ruelle operators established in [LZ24c]. This requires checking a strong non-integrability condition (Definition 2.13), which in particular implies that the potential is not cohomologous to a constant. For the bounded part, we employ Ruelle's estimate (see Appendix A). For the local part, we employ the complex Ruelle–Perron–Frobenius theorem [Pol84, Theorem 2] and arguments in [PP90].

Finally, to count orbits in intervals I_n, we approximate the indicator function $\mathbb{1}_{I_n}$ by smooth test functions. We then apply Fourier transforms to relate the smoothed count to partition functions, which allows us to apply the established decay estimates.

The paper is organized as follows. In Section 2, we fix our notation, review fundamental concepts from thermodynamic formalism, and recall key results from the theory of expanding Thurston maps. In Section 3, we collect the main assumptions used throughout the paper. The technical core of the paper is Section 4, where we employ previous results and derive crucial decay estimates for associated partition functions. These estimates are then used in Section 5 to prove Theorem 1.1.

2. Preliminaries

2.1. **Notation.** Let $\mathbb{C}$ be the complex plane and $\widehat{\mathbb{C}}$ be the Riemann sphere. For each complex number $z \in \mathbb{C}$, we denote by $\mathrm{Re}(z)$ the real part of z, and by $\mathrm{Im}(z)$ the imaginary part of z. The symbol $\mathbf{i}$ stands for the imaginary unit in the complex plane $\mathbb{C}$. We follow the convention that $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. We denote by $\mathbb{R}_{\geqslant 0}$ the set of non-negative real numbers. The cardinality of a set A is denoted by $\mathrm{card}(A)$.

Consider complex-valued functions u, v, and w defined on $\mathbb{R}$ and $a \in \mathbb{R} \cup \{\pm\infty\}$. We write $u(x) \sim v(x)$ as $x \to a$ if $\lim_{x\to a} \frac{u(x)}{v(x)} = 1$, and write $u(x) = v(x) + \mathcal{O}(w(x))$ as $x \to a$ if $\limsup_{x\to a} \big|\frac{u(x)-v(x)}{w(x)}\big| < +\infty$. We use the same notation for discrete variables.

Let $f\colon X \to X$ be a map on a set X. The inverse map of f is denoted by f^{-1}. We write f^n for the n-th iterate of f, and $f^{-n} := (f^n)^{-1}$, for $n \in \mathbb{N}$. For $n \in \mathbb{N}$, we denote by

$$\operatorname{Fix}(f^n) := \{x \in X : f^n(x) = x\}$$

the set of fixed points of f^n. For a real-valued function $\varphi\colon X \to \mathbb{R}$, we write

$$S_n\varphi(x) = S_n^f\varphi(x) := \sum_{j=0}^{n-1} \varphi(f^j(x))$$

for $x \in X$ and $n \in \mathbb{N}_0$. We omit the superscript f when the map f is clear from the context. We adopt the convention that $S_0\varphi = 0$.

Let (X, d) be a compact metric space. For each subset $Y \subseteq X$, we denote the diameter of Y by $\operatorname{diam}_d(Y) := \sup\{d(x, y) : x,\, y \in Y\}$ and the characteristic function of Y by $\mathbb{1}_Y$. We denote by $C(X)$ (resp. $C(X, \mathbb{C})$) the space of real-valued continuous functions from X to $\mathbb{R}$ (resp. complex-valued continuous functions from X to $\mathbb{C}$). For $\varphi \in C(X)$, we denote by $\operatorname{supp}\varphi$ the support of φ. If we do not specify otherwise, we equip $C(X)$ and $C(X, \mathbb{C})$ with the uniform norm $\|\cdot\|_\infty$. For $\psi \in C(X, \mathbb{C})$, we denote

$$|\psi|_\beta := \sup\bigl\{|\psi(x) - \psi(y)|/d(x, y)^\beta : x, y \in X,\, x \neq y\bigr\},$$

and the Hölder norm of ψ is defined as $\|\psi\|_{C^{0,\beta}} := |\psi|_\beta + \|\psi\|_\infty$. The space of real-valued (resp. complex-valued) Hölder continuous functions with an exponent $\beta \in (0, 1]$ on (X, d) is denoted by $C^{0,\beta}(X, d)$ (resp. $C^{0,\beta}((X, d), \mathbb{C})$), which consists of continuous functions with finite Hölder norm.

2.2. **Thermodynamic formalism.** We first review some basic concepts from ergodic theory and dynamical systems. For more detailed studies of these concepts, we refer the reader to [KH95, Chapter 20] and [Wal82, Chapter 9].

Let (X, d) be a compact metric space and $g\colon X \to X$ a continuous map. Given $n \in \mathbb{N}$,

$$d_g^n(x, y) := \max\bigl\{d\bigl(g^k(x), g^k(y)\bigr) : k \in \{0,\, 1,\, \dots,\, n-1\}\bigr\}, \qquad \text{for } x,\, y \in X,$$

defines a metric on X. A set $F \subseteq X$ is (n, ϵ)*-separated* (with respect to g), for some $n \in \mathbb{N}$ and $\epsilon > 0$, if for each pair of distinct points $x,\, y \in F$, we have $d_g^n(x, y) \geqslant \epsilon$.

For each real-valued continuous function $\psi \in C(X)$, the following limits exist and are equal, and we denote these limits by $P(g, \psi)$ (see e.g. [KH95, Subsection 20.2]):

$$P(g, \psi) := \lim_{\epsilon \to 0^+} \limsup_{n \to +\infty} \frac{1}{n} \log N_d(g, \psi, \varepsilon, n) = \lim_{\epsilon \to 0^+} \liminf_{n \to +\infty} \frac{1}{n} \log N_d(g, \psi, \varepsilon, n), \tag{2.1}$$

where

$$N_d(g, \psi, \varepsilon, n) := \sup\Bigl\{\sum_{x \in E} \exp\bigl(S_n\psi(x)\bigr) : E \subseteq X \text{ is } (n, \varepsilon)\text{-separated with respect to } g\Bigr\}.$$

We call $P(g, \psi)$ the *topological pressure* of g with respect to the *potential* ψ. The quantity $h_{\mathrm{top}}(g) := P(g, 0)$ is called the *topological entropy* of g. The topological pressure $P(g, \psi)$ depends only on the topology of X (see e.g. [KH95, Subsection 20.2]).

Let $\mathcal{M}(X, g)$ be the set of g-invariant Borel probability measures on X. Let $\mu \in \mathcal{M}(X, g)$. We say g is *ergodic* for μ (or μ is *ergodic* for g) if for each set $A \in \mathcal{B}$ with $g^{-1}(A) = A$ we have $\mu(A) = 0$ or $\mu(A) = 1$.

For each real-valued continuous function $\psi \in C(X)$, the *measure-theoretic pressure* $P_\mu(g, \psi)$ of g for the measure $\mu \in \mathcal{M}(X, g)$ and the potential ψ is

$$P_\mu(g, \psi) := h_\mu(g) + \int \psi \,\mathrm{d}\mu, \tag{2.2}$$

where $h_\mu(g)$ is the measure-theoretic entropy of g for μ.

The topological pressure is related to the measure-theoretic pressure by the so-called *Variational Principle*. It states that (see e.g. [KH95, Theorem 20.2.4])

$$P(g,\psi) = \sup\{P_\mu(g,\psi) : \mu \in \mathcal{M}(X,g)\} \tag{2.3}$$

for each $\psi \in C(X)$. In particular, when ψ is the constant function 0,

$$h_{\mathrm{top}}(g) = \sup\{h_\mu(g) : \mu \in \mathcal{M}(X,g)\}. \tag{2.4}$$

A measure μ that attains the supremum in (2.3) is called an *equilibrium state* for the map g and the potential ψ. A measure μ that attains the supremum in (2.4) is called a *measure of maximal entropy* of g.

Let $\widetilde{X}$ be another compact metric space. If μ is a measure on X and the map $\pi\colon X \to \widetilde{X}$ is continuous, then the *push-forward* $\pi_*\mu$ of μ by π is the measure given by $\pi_*\mu(A) \coloneqq \mu\bigl(\pi^{-1}(A)\bigr)$ for all Borel sets $A \subseteq \widetilde{X}$.

2.3. **Thurston maps.** In this subsection, we go over some key concepts and results on Thurston maps, and expanding Thurston maps in particular. For a more thorough treatment of the subject, we refer to [BM17, Li17].

Let S^2 denote an oriented topological 2-sphere and $f\colon S^2 \to S^2$ be a branched covering map. We denote by $\deg_f(x)$ the local degree of f at $x \in S^2$. The *degree* of f is $\deg f = \sum_{x\in f^{-1}(y)} \deg_f(x)$ for $y \in S^2$ and is independent of y.

A point $x \in S^2$ is a *critical point* of f if $\deg_f(x) \geqslant 2$. The set of critical points of f is denoted by $\operatorname{crit} f$. A point $y \in S^2$ is a *postcritical point* of f if $y = f^n(x)$ for some $x \in \operatorname{crit} f$ and $n \in \mathbb{N}$. The set of postcritical points of f is denoted by $\operatorname{post} f$. We observe that $\operatorname{post} f = \operatorname{post} f^n$ for all $n \in \mathbb{N}$.

Definition 2.1 (Thurston maps)**.** A Thurston map is a branched covering map $f\colon S^2 \to S^2$ on S^2 with $\deg f \geqslant 2$ and $\operatorname{card}(\operatorname{post} f) < +\infty$.

We now recall the notation for cell decompositions of S^2 as used in [BM17] and [Li17]. A *cell of dimension* n in S^2, $n \in \{1, 2\}$, is a subset $c \subseteq S^2$ that is homeomorphic to the closed unit ball $\overline{\mathbb{B}^n}$ in $\mathbb{R}^n$, where $\mathbb{B}^n$ is the open unit ball in $\mathbb{R}^n$. We define the *boundary of* c, denoted by ∂c, to be the set of points corresponding to $\partial\mathbb{B}^n$ under such a homeomorphism between c and $\overline{\mathbb{B}^n}$. The *interior of* c is defined to be $\operatorname{inte}(c) = c \smallsetminus \partial c$. For each point $x \in S^2$, the set $\{x\}$ is considered as a *cell of dimension* 0 in S^2. For a cell c of dimension 0, we adopt the convention that $\partial c = \emptyset$ and $\operatorname{inte}(c) = c$.

Let $f\colon S^2 \to S^2$ be a Thurston map, and $\mathcal{C} \subseteq S^2$ be a Jordan curve containing $\operatorname{post} f$. Then the pair f and $\mathcal{C}$ induces natural cell decompositions $\mathbf{D}^n(f,\mathcal{C})$ of S^2, for each $n \in \mathbb{N}_0$, in the following way:

By the Jordan curve theorem, the set $S^2 \smallsetminus \mathcal{C}$ has two connected components. We call the closure of one of them the *white* 0*-tile* for $(f,\mathcal{C})$, denoted by $X^0_{\mathfrak{w}}$, and the closure of the other one the *black* 0*-tile* for $(f,\mathcal{C})$, denoted be $X^0_{\mathfrak{b}}$. The set of 0*-tiles* is $\mathbf{X}^0(f,\mathcal{C}) \coloneqq \bigl\{X^0_{\mathfrak{b}}, X^0_{\mathfrak{w}}\bigr\}$. The set of 0*-vertices* is $\mathbf{V}^0(f,\mathcal{C}) \coloneqq \operatorname{post} f$. We set $\overline{\mathbf{V}}^0(f,\mathcal{C}) \coloneqq \bigl\{\{x\} : x \in \mathbf{V}^0(f,\mathcal{C})\bigr\}$. The set of 0*-edges* $\mathbf{E}^0(f,\mathcal{C})$ consists of the closures of the connected components of $\mathcal{C} \smallsetminus \operatorname{post} f$. Then we get a cell decomposition

$$\mathbf{D}^0(f,\mathcal{C}) \coloneqq \mathbf{X}^0(f,\mathcal{C}) \cup \mathbf{E}^0(f,\mathcal{C}) \cup \overline{\mathbf{V}}^0(f,\mathcal{C})$$

of S^2 consisting of *cells of level* 0, or 0*-cells*.

We can recursively define the unique cell decomposition $\mathbf{D}^n(f,\mathcal{C})$ for $n \in \mathbb{N}$, consisting of n*-cells*, such that f is cellular for $\bigl(\mathbf{D}^{n+1}(f,\mathcal{C}), \mathbf{D}^n(f,\mathcal{C})\bigr)$. See [BM17, Lemma 5.12] for details. We denote by $\mathbf{X}^n(f,\mathcal{C})$ the set of n-cells of dimension 2, called n*-tiles*; by $\mathbf{E}^n(f,\mathcal{C})$ the set of n-cells of dimension 1, called n*-edges*; by $\overline{\mathbf{V}}^n(f,\mathcal{C})$ the set of n-cells of dimension 0; and by $\mathbf{V}^n(f,\mathcal{C})$ the set $\bigl\{x : \{x\} \in \overline{\mathbf{V}}^n(f,\mathcal{C})\bigr\}$, called the set of n*-vertices*.

For $n \in \mathbb{N}_0$, we define the *set of black* n*-tiles* as

$$\mathbf{X}^n_{\mathfrak{b}}(f,\mathcal{C}) \coloneqq \bigl\{X \in \mathbf{X}^n(f,\mathcal{C}) : f^n(X) = X^0_{\mathfrak{b}}\bigr\},$$

and the *set of white n-tiles* as

$$\mathbf{X}^n_{\mathfrak{w}}(f,\mathcal{C}) := \left\{X \in \mathbf{X}^n(f,\mathcal{C}) : f^n(X) = X^0_{\mathfrak{w}}\right\}.$$

From now on, if the map f and the Jordan curve $\mathcal{C}$ are clear from the context, we will sometimes omit $(f,\mathcal{C})$ in the notation above.

We can now give a definition of expanding Thurston maps.

Definition 2.2 (Expansion)**.** A Thurston map $f\colon S^2 \to S^2$ is called *expanding* if there exists a metric d on S^2 that induces the standard topology on S^2 and a Jordan curve $\mathcal{C} \subseteq S^2$ containing $\operatorname{post} f$ such that

$$\lim_{n\to+\infty} \max\{\operatorname{diam}_d(X) : X \in \mathbf{X}^n(f,\mathcal{C})\} = 0. \tag{2.5}$$

For an expanding Thurston map f, we can fix a particular metric d on S^2 called a *visual metric for f*. For the existence and properties of such metrics, see [BM17, Chapter 8]. For a visual metric d for f, there exists a unique constant $\Lambda > 1$ called the *expansion factor* of d (see [BM17, Chapter 8] for more details).

Remark 2.3. If $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational expanding Thurston map, then a visual metric is quasisymmetrically equivalent to the chordal metric on the Riemann sphere $\widehat{\mathbb{C}}$ (see [BM17, Theorem 18.1 (ii)]). Here the chordal metric σ on $\widehat{\mathbb{C}}$ is given by $\sigma(z,w) := \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}$ for all $z, w \in \mathbb{C}$, and $\sigma(\infty,z) = \sigma(z,\infty) := \frac{2}{\sqrt{1+|z|^2}}$ for all $z \in \mathbb{C}$. Quasisymmetric embeddings of bounded connected metric spaces are Hölder continuous (see [Hei01, Section 11.1 and Corollary 11.5]). Accordingly, the classes of Hölder continuous functions on $\widehat{\mathbb{C}}$ equipped with the chordal metric and on $S^2 = \widehat{\mathbb{C}}$ equipped with any visual metric for f are the same (up to a change of the Hölder exponent).

A Jordan curve $\mathcal{C} \subseteq S^2$ is *f-invariant* if $f(\mathcal{C}) \subseteq \mathcal{C}$. We are interested in f-invariant Jordan curves that contain $\operatorname{post} f$, since for such a Jordan curve $\mathcal{C}$, we get a cellular Markov partition $\left(\mathbf{D}^1(f,\mathcal{C}), \mathbf{D}^0(f,\mathcal{C})\right)$ for f. According to Example 15.11 in [BM17], such f-invariant Jordan curves containing $\operatorname{post} f$ need not exist. However, Bonk and Meyer [BM17, Theorem 15.1] proved that there exists an f^n-invariant Jordan curve $\mathcal{C}$ containing $\operatorname{post} f$ for each sufficiently large n depending on f. A slightly stronger version of this result was proved in [Li16, Lemma 3.11], and we record it below.

Lemma 2.4 (Bonk & Meyer [BM17]; Li [Li16])**.** *Let $f\colon S^2 \to S^2$ be an expanding Thurston map, and $\widetilde{\mathcal{C}} \subseteq S^2$ be a Jordan curve with $\operatorname{post} f \subseteq \widetilde{\mathcal{C}}$. Then there exists an integer $N(f,\widetilde{\mathcal{C}}) \in \mathbb{N}$ such that for each $n \geqslant N(f,\widetilde{\mathcal{C}})$ there exists an f^n-invariant Jordan curve $\mathcal{C}$ isotopic to $\widetilde{\mathcal{C}}$ rel. $\operatorname{post} f$ such that no n-tile in $\mathbf{X}^n(f,\mathcal{C})$ joins opposite sides of $\mathcal{C}$.*

The phrase "joining opposite sides" has a specific meaning in our context.

Definition 2.5 (Joining opposite sides)**.** Fix a Thurston map f with $\operatorname{card}(\operatorname{post} f) \geqslant 3$ and an f-invariant Jordan curve $\mathcal{C}$ containing $\operatorname{post} f$. A set $K \subseteq S^2$ *joins opposite sides* of $\mathcal{C}$ if K meets two disjoint 0-edges when $\operatorname{card}(\operatorname{post} f) \geqslant 4$, or K meets all three 0-edges when $\operatorname{card}(\operatorname{post} f) = 3$.

Recall that $\operatorname{card}(\operatorname{post} f) \geqslant 3$ for each expanding Thurston map f [BM17, Lemma 6.1].

2.4. **Symbolic dynamics for expanding Thurston maps.** In this subsection, we briefly review the dynamics of one-sided subshifts of finite type. We refer the reader to [Kit98] for a beautiful introduction to symbolic dynamics. For a discussion of results on subshifts of finite type in our context, see [PP90, Bal00].

Let S be a finite non-empty set and $A\colon S \times S \to \{0,\,1\}$ be a matrix whose entries are either 0 or 1. We denote the *set of admissible sequences defined by A* by

$$\Sigma_A^+ := \{\{x_i\}_{i\in\mathbb{N}_0} : x_i \in S,\, A(x_i,x_{i+1}) = 1, \text{ for each } i \in \mathbb{N}_0\}.$$

Given $\tau \in (0,1)$, we equip the set Σ_A^+ with a metric d_τ given by

$$d_\tau(\{x_i\}_{i\in\mathbb{N}_0}, \{y_i\}_{i\in\mathbb{N}_0}) = \tau^m \qquad \text{for } \{x_i\}_{i\in\mathbb{N}_0} \neq \{y_i\}_{i\in\mathbb{N}_0}, \tag{2.6}$$

where m is the smallest non-negative integer such that $x_m \neq y_m$. The topology on the metric space (Σ_A^+, d_τ) coincides with that induced from the product topology, and is therefore compact.

The *left-shift operator* $\sigma_A \colon \Sigma_A^+ \to \Sigma_A^+$ (defined by A) is given by

$$\sigma_A(\{x_i\}_{i\in\mathbb{N}_0}) = \{x_{i+1}\}_{i\in\mathbb{N}_0} \qquad \text{for } \{x_i\}_{i\in\mathbb{N}_0} \in \Sigma_A^+.$$

The pair (Σ_A^+, σ_A) is called the *one-sided subshift of finite type* defined by A. The set S is called the *set of states* and the matrix $A \colon S \times S \to \{0, 1\}$ is called the *transition matrix*. In particular, if all entries of the transition matrix A are 1, we call $\sigma \colon \Sigma^+ \to \Sigma^+$ a *one-sided shift map* on $\operatorname{card}(S)$ symbols, where $\sigma = \sigma_A$ and $\Sigma^+ = \Sigma_A^+ = S^{\mathbb{N}_0}$.

For a complex-valued continuous function $\psi \in C(\Sigma_A^+, \mathbb{C})$, the *Ruelle operator* $\mathcal{L}_\psi \colon C(\Sigma_A^+, \mathbb{C}) \to C(\Sigma_A^+, \mathbb{C})$ is defined by

$$(\mathcal{L}_\psi u)(x) \coloneqq \sum_{y\in\sigma_A^{-1}(x)} e^{\psi(y)} u(y) \tag{2.7}$$

for $u \in C(\Sigma_A^+, \mathbb{C})$ and $x \in \Sigma_A^+$.

We say that a one-sided subshift of finite type (Σ_A^+, σ_A) is *topologically mixing* if there exists $N \in \mathbb{N}$ such that $A^n(x,y) > 0$ for all $n \geqslant N$ and $x, y \in S$.

Let X and Y be topological spaces, and $G \colon X \to X$ and $g \colon Y \to Y$ be continuous maps. We say that the topological dynamical system (Y, g) is a *factor* of the topological dynamical system (X, G) if there is a surjective continuous map $\pi \colon X \to Y$ such that $\pi \circ G = g \circ \pi$. Such a map $\pi \colon X \to Y$ is called a *semi-conjugacy* or *factor map*.

The following proposition, combining results from [LZ24a, Propositions 3.31 and 5.5], constructs a Hölder continuous semi-conjugacy from a one-sided subshift of finite type to an expanding Thurston map, using a coding based on tiles.

Proposition 2.6 (Li & Zheng [LZ24a])**.** *Let $f \colon S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $f(\mathcal{C}) \subseteq \mathcal{C}$ and* $\operatorname{post} f \subseteq \mathcal{C}$. *Let d be a visual metric on S^2 for f. We set $S_\triangle \coloneqq \mathbf{X}^1(f, \mathcal{C})$, and define a transition matrix $A_\triangle \colon S_\triangle \times S_\triangle \to \{0, 1\}$ by*

$$A_\triangle(X, X') \coloneqq \begin{cases} 1 & \textit{if } f(X) \supseteq X', \\ 0 & \textit{otherwise} \end{cases}$$

for X, $X' \in S_\triangle$. Then the dynamical system (S^2, f) is a factor of the one-sided subshift of finite type $(\Sigma_{A_\triangle}^+, \sigma_{A_\triangle})$ defined by the transition matrix $A_\triangle$ with a surjective and Hölder continuous factor map $\pi_\triangle \colon \Sigma_{A_\triangle}^+ \to S^2$ given by

$$\pi_\triangle(\{X_i\}_{i\in\mathbb{N}_0}) = x, \quad \textit{where } \{x\} = \bigcap_{i\in\mathbb{N}_0} f^{-i}(X_i). \tag{2.8}$$

Here $\Sigma_{A_\triangle}^+$ is equipped with the metric d_τ defined in (2.6) *for some constant $\tau \in (0,1)$, and S^2 is equipped with the visual metric d. Moreover, $(\Sigma_{A_\triangle}^+, \sigma_{A_\triangle})$ is topologically mixing, $\pi_\triangle$ is injective on $\pi_\triangle^{-1}\big(S^2 \smallsetminus \bigcup_{i\in\mathbb{N}_0} f^{-i}(\mathcal{C})\big)$, and $P(\sigma_{A_\triangle}, \psi \circ \pi_\triangle) = P(f, \psi)$ for each $\psi \in C^{0,\beta}(S^2, d)$.*

Remark 2.7. By [Li18, Lemma 5.10], the transition matrix $A_\triangle$ in Proposition 2.6 is primitive. Recall that a matrix A is called primitive if there exists $n \in \mathbb{N}$ such that all entries of A^n are positive.

The symbolic model in Proposition 2.6 is based on coding by tiles. We next recall from [LZ24a] the construction of two subshifts of finite type based on the dynamics on an invariant Jordan curve.

Let $f\colon S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $f(\mathcal{C}) \subseteq \mathcal{C}$ and $\operatorname{post} f \subseteq \mathcal{C}$. Define the set of states $\mathcal{S}_{\mathrm{I}} := \{e \in \mathbf{E}^1(f,\mathcal{C}) : e \subseteq \mathcal{C}\}$ and the transition matrix $A_{\mathrm{I}}\colon \mathcal{S}_{\mathrm{I}} \times \mathcal{S}_{\mathrm{I}} \to \{0,1\}$ by

$$A_{\mathrm{I}}(e_1, e_2) = \begin{cases} 1 & \text{if } f(e_1) \supseteq e_2; \\ 0 & \text{otherwise} \end{cases} \tag{2.9}$$

for $e_1, e_2 \in \mathcal{S}_{\mathrm{I}}$.

Define the set of states $\mathcal{S}_{\mathrm{II}} := \bigl\{(e,c) \in \mathbf{E}^1(f,\mathcal{C}) \times \{\mathfrak{b}, \mathfrak{w}\} : e \subseteq \mathcal{C}\bigr\}$. For each $(e,c) \in \mathcal{S}_{\mathrm{II}}$, we denote by $X^1(e,c) \in \mathbf{X}^1(f,\mathcal{C})$ the unique 1-tile satisfying[1]

$$e \subseteq X^1(e,c) \subseteq X^0_c. \tag{2.10}$$

We define the transition matrix $A_{\mathrm{II}}\colon \mathcal{S}_{\mathrm{II}} \times \mathcal{S}_{\mathrm{II}} \to \{0,1\}$ by

$$A_{\mathrm{II}}((e_1,c_1),(e_2,c_2)) = \begin{cases} 1 & \text{if } f(e_1) \supseteq e_2 \text{ and } f\bigl(X^1(e_1,c_1)\bigr) \supseteq X^1(e_2,c_2); \\ 0 & \text{otherwise} \end{cases}$$

for $(e_1,c_1),(e_2,c_2) \in \mathcal{S}_{\mathrm{II}}$.

We will consider the one-sided subshift of finite type $\bigl(\Sigma^+_{A_{\mathrm{I}}}, \sigma_{A_{\mathrm{I}}}\bigr)$ defined by the transition matrix A_{I}, and $\bigl(\Sigma^+_{A_{\mathrm{II}}}, \sigma_{A_{\mathrm{II}}}\bigr)$ defined by the transition matrix A_{II}, where

$$\begin{aligned} \Sigma^+_{A_{\mathrm{I}}} &= \{\{e_i\}_{i\in\mathbb{N}_0} : e_i \in \mathcal{S}_{\mathrm{I}} \text{ and } A_{\mathrm{I}}(e_i, e_{i+1}) = 1 \text{ for each } i \in \mathbb{N}_0\}, \\ \Sigma^+_{A_{\mathrm{II}}} &= \{\{(e_i,c_i)\}_{i\in\mathbb{N}_0} : (e_i,c_i) \in \mathcal{S}_{\mathrm{II}} \text{ and } A_{\mathrm{II}}((e_i,c_i),(e_{i+1},c_{i+1})) = 1 \text{ for each } i \in \mathbb{N}_0\}, \end{aligned}$$

and the maps $\sigma_{A_{\mathrm{I}}}\colon \Sigma^+_{A_{\mathrm{I}}} \to \Sigma^+_{A_{\mathrm{I}}}$ and $\sigma_{A_{\mathrm{II}}}\colon \Sigma^+_{A_{\mathrm{II}}} \to \Sigma^+_{A_{\mathrm{II}}}$ are the corresponding left-shift operators.

The following proposition from [LZ24a, Proposition 6.1] shows that the dynamics of f on the invariant Jordan curve $\mathcal{C}$ is a factor of the edge-based symbolic dynamics defined above.

Proposition 2.8 (Li & Zheng [LZ24a])**.** *Let $f\colon S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $f(\mathcal{C}) \subseteq \mathcal{C}$ and $\operatorname{post} f \subseteq \mathcal{C}$. Let d be a visual metric on S^2 for f. Let $\bigl(\Sigma^+_{A_\Delta}, \sigma_{A_\Delta}\bigr)$ be the one-sided subshift of finite type associated to f and $\mathcal{C}$ defined in Proposition 2.6, and let $\pi_\Delta\colon \Sigma^+_{A_\Delta} \to S^2$ be the factor map defined in* (2.8)*. Fix $\tau \in (0,1)$ and equip the spaces $\Sigma^+_{A_{\mathrm{I}}}$ and $\Sigma^+_{A_{\mathrm{II}}}$ with the metric d_τ defined in* (2.6)*. We write $\mathbf{V}(f,\mathcal{C}) := \bigcup_{i\in\mathbb{N}_0} \mathbf{V}^i(f,\mathcal{C})$. Then the following statements hold:*

(i) *$\bigl(\Sigma^+_{A_{\mathrm{I}}}, \sigma_{A_{\mathrm{I}}}\bigr)$ is a factor of $\bigl(\Sigma^+_{A_{\mathrm{II}}}, \sigma_{A_{\mathrm{II}}}\bigr)$ with a Lipschitz continuous factor map $\pi_{\mathrm{II}}\colon \Sigma^+_{A_{\mathrm{II}}} \to \Sigma^+_{A_{\mathrm{I}}}$ defined by*

$$\pi_{\mathrm{II}}(\{(e_i,c_i)\}_{i\in\mathbb{N}_0}) = \{e_i\}_{i\in\mathbb{N}_0} \tag{2.11}$$

for $\{(e_i,c_i)\}_{i\in\mathbb{N}_0} \in \Sigma^+_{A_{\mathrm{II}}}$. Moreover, for each $\{e_i\}_{i\in\mathbb{N}_0} \in \Sigma^+_{A_{\mathrm{I}}}$, we have

$$\operatorname{card}\bigl(\pi_{\mathrm{II}}^{-1}(\{e_i\}_{i\in\mathbb{N}_0})\bigr) = 2.$$

(ii) *$(\mathcal{C}, f|_{\mathcal{C}})$ is a factor of $\bigl(\Sigma^+_{A_{\mathrm{I}}}, \sigma_{A_{\mathrm{I}}}\bigr)$ with a Hölder continuous factor map $\pi_{\mathrm{I}}\colon \Sigma^+_{A_{\mathrm{I}}} \to \mathcal{C}$ defined by*

$$\pi_{\mathrm{I}}(\{e_i\}_{i\in\mathbb{N}_0}) = x, \quad \textit{where } \{x\} = \bigcap_{i\in\mathbb{N}_0} f^{-i}(e_i) \tag{2.12}$$

for $\{e_i\}_{i\in\mathbb{N}_0} \in \Sigma^+_{A_{\mathrm{I}}}$. Moreover, for each $x \in \mathcal{C}$, we have

$$\operatorname{card}\bigl(\pi_{\mathrm{I}}^{-1}(x)\bigr) = \begin{cases} 1 & \textit{if } x \in \mathcal{C} \setminus \mathbf{V}(f,\mathcal{C}), \\ 2 & \textit{if } x \in \mathcal{C} \cap \mathbf{V}(f,\mathcal{C}). \end{cases} \tag{2.13}$$

---

[1] The existence and uniqueness of such a tile $X^1(e,c)$ defined by (2.10) follow immediately from the properties of the cell decomposition (cf. [BM17, Proposition 5.16 (iii), (v), and (vi)]) and the assumptions that $f(\mathcal{C}) \subseteq \mathcal{C}$ and $e \subseteq \mathcal{C}$.

Thus, we have the following commutative diagram:

$$
\begin{tikzcd}
\Sigma_{A_{\shortparallel}}^{+} \arrow[r, "\pi_{\shortparallel}"] \arrow[d, "\sigma_{A_{\shortparallel}}"] & \Sigma_{A_{\shortmid}}^{+} \arrow[r, "\pi_{\shortmid}"] \arrow[d, "\sigma_{A_{\shortmid}}"] & \mathcal{C} \arrow[d, "f|_{\mathcal{C}}"] \\
\Sigma_{A_{\shortparallel}}^{+} \arrow[r, "\pi_{\shortparallel}"] & \Sigma_{A_{\shortmid}}^{+} \arrow[r, "\pi_{\shortmid}"] & \mathcal{C}.
\end{tikzcd}
$$

2.5. **Ergodic theory of expanding Thurston maps.** We summarize the existence, uniqueness, and basic properties of equilibrium states for expanding Thurston maps in the following theorem.

Theorem 2.9 (Li [Li18])**.** *Let $f\colon S^2 \to S^2$ be an expanding Thurston map and d a visual metric on S^2 for f. Let $\phi, \gamma \in C^{0,\beta}\bigl(S^2, d\bigr)$ be real-valued Hölder continuous functions with an exponent $\beta \in (0,1]$. Then the following statements hold:*

(i) *There exists a unique equilibrium state μ_ϕ for the map f and the potential ϕ.*

(ii) *For each $t \in \mathbb{R}$, we have $\frac{\mathrm{d}}{\mathrm{d}t} P(f, \phi + t\gamma) = \int \gamma \,\mathrm{d}\mu_{\phi+t\gamma}$.*

(iii) *Let μ_γ be the unique equilibrium state for f and γ. Then $\mu_\phi = \mu_\gamma$ if and only if there exist a constant $K \in \mathbb{R}$ and a continuous function $u \in C\bigl(S^2\bigr)$ such that $\phi - \gamma = K + u \circ f - u$.*

Let $T\colon X \to X$ be a map on a topological space X and $\psi\colon X \to \mathbb{R}$ be a real-valued function on X. We say that ψ is *cohomologous to a constant in* $C(X)$ if there exist $C \in \mathbb{R}$ and $u \in C(X)$ such that $\psi = C + u \circ T - u$.

Lemma 2.10. *Under the assumptions and notation of Proposition 2.6, let $\phi \in C^{0,\beta}\bigl(S^2, d\bigr)$ and $\widetilde{\mu} \in \mathcal{M}\bigl(\Sigma_{A_\vartriangle}^{+}, \sigma_{A_\vartriangle}\bigr)$ be the equilibrium state for the map $\sigma_{A_\vartriangle}$ and the potential $\phi \circ \pi_\vartriangle$. Denote $\mu \coloneqq (\pi_\vartriangle)_*\widetilde{\mu}$. Then $h_\mu(f) = h_{\widetilde{\mu}}(\sigma_{A_\vartriangle})$ and μ is an equilibrium state for f and ϕ. Moreover, ϕ is cohomologous to a constant in $C\bigl(S^2\bigr)$ with respect to f if and only if $\phi \circ \pi_\vartriangle$ is cohomologous to a constant in $C\bigl(\Sigma_{A_\vartriangle}^{+}\bigr)$ with respect to $\sigma_{A_\vartriangle}$.*

Proof. Let $E \coloneqq \bigcup_{i \in \mathbb{N}_0} f^{-i}(\mathcal{C})$ and $\widetilde{E} \coloneqq \pi_\vartriangle^{-1}(E)$. To deduce that $h_\mu(f) = h_{\widetilde{\mu}}(\sigma_{A_\vartriangle})$, it suffices to show $\mu(E) = \widetilde{\mu}(\widetilde{E}) = 0$. Indeed, by Proposition 2.6, the restriction of $\pi_\vartriangle$ to the subset $\pi_\vartriangle^{-1}\bigl(S^2 \smallsetminus E\bigr) = \Sigma_{A_\vartriangle}^{+} \smallsetminus \widetilde{E}$ is a bijection from $\Sigma_{A_\vartriangle}^{+} \smallsetminus \widetilde{E}$ onto $S^2 \smallsetminus E$. If $\mu(E) = \widetilde{\mu}(\widetilde{E}) = 0$, then $\pi_\vartriangle$ is a measurable isomorphism between the systems $\bigl(\Sigma_{A_\vartriangle}^{+}, \sigma_{A_\vartriangle}, \widetilde{\mu}\bigr)$ and $\bigl(S^2, f, \mu\bigr)$ (modulo sets of measure zero). Since measure-theoretic entropy is invariant under measurable isomorphism, it follows that $h_\mu(f) = h_{\widetilde{\mu}}(\sigma_{A_\vartriangle})$.

We argue by contradiction and suppose that $\mu(E) > 0$. The equilibrium state $\widetilde{\mu}$ for the topologically mixing subshift $\bigl(\Sigma_{A_\vartriangle}^{+}, \sigma_{A_\vartriangle}\bigr)$ is ergodic (cf. [Bow75, Proposition 1.14]). Since $\pi_\vartriangle$ is a factor map, the push-forward measure μ is also ergodic. Since $f(\mathcal{C}) \subseteq \mathcal{C}$, we have $f^{-1}(E) = E$. Then ergodicity implies $\mu(E) = 1$. Furthermore, we can show that $\mu(\mathcal{C}) = 1$. Indeed, since μ is f-invariant and $f(\mathcal{C}) \subseteq \mathcal{C}$, we have $\mu(f^{-n}(\mathcal{C})) = \mu(\mathcal{C})$ and $\mathcal{C} \subseteq f^{-n}(\mathcal{C})$ for all $n \in \mathbb{N}_0$, which imply that $\mu(f^{-n}(\mathcal{C}) \smallsetminus \mathcal{C}) = 0$. Thus $\mu(E \smallsetminus \mathcal{C}) = 0$, i.e., $\mu(\mathcal{C}) = 1$. Consequently, $\widetilde{\mu}\bigl(\pi_\vartriangle^{-1}(\mathcal{C})\bigr) = 1$. This leads to a contradiction, as $\pi_\vartriangle^{-1}(\mathcal{C})$ is a closed proper subset of $\Sigma_{A_\vartriangle}^{+}$, while the measure $\widetilde{\mu}$ must be positive on every non-empty open set (as a consequence of the Gibbs property [Bow75, Theorem 1.2]). Therefore, we conclude that $0 = \mu(E) = \widetilde{\mu}\bigl(\widetilde{E}\bigr)$.

We now verify that μ is an equilibrium state for f and ϕ. Since $\mu = (\pi_\vartriangle)_*\widetilde{\mu}$, we have $\int \phi \,\mathrm{d}\mu = \int \phi \circ \pi_\vartriangle \,\mathrm{d}\widetilde{\mu}$. Using the entropy preservation $h_\mu(f) = h_{\widetilde{\mu}}(\sigma_{A_\vartriangle})$ and the pressure equality $P(\sigma_{A_\vartriangle}, \phi \circ \pi_\vartriangle) = P(f, \phi)$ from Proposition 2.6, we obtain that

$$
P(f, \phi) = P(\sigma_{A_\vartriangle}, \phi \circ \pi_\vartriangle) = h_{\widetilde{\mu}}(\sigma_{A_\vartriangle}) + \int \phi \circ \pi_\vartriangle \,\mathrm{d}\widetilde{\mu} = h_\mu(f) + \int \phi \,\mathrm{d}\mu.
$$

This implies that μ is an equilibrium state for f and ϕ.

Finally, we establish the equivalence of the cohomological statements. The forward direction is straightforward. For the converse, assume that $\phi \circ \pi_\vartriangle$ is cohomologous to a constant in $C\big(\Sigma_{A_\vartriangle}^+\big)$. By Proposition 2.6, the subshift $\big(\Sigma_{A_\vartriangle}^+, \sigma_{A_\vartriangle}\big)$ is topologically mixing. It is a classical result (see [Bow75, Theorem 1.28]) that for such a system, the equilibrium state of a potential cohomologous to a constant is the measure of maximal entropy. Thus, $\widetilde{\mu}$ is the measure of maximal entropy for $\sigma_{A_\vartriangle}$. The preservation of entropy implies that its pushforward $\mu = (\pi_\vartriangle)_*\tilde{\mu}$ is a measure of maximal entropy for f. Since μ is an equilibrium state for f and ϕ, it follows from Theorem 2.9 (iii) that ϕ is cohomologous to a constant in $C\big(S^2\big)$. This completes the proof. $\square$

The potentials that satisfy the following property are of particular interest in the considerations of Prime Orbit Theorems.

Definition 2.11 (Eventually positive function)**.** Let $g\colon X \to X$ be a map on a set X, and $\varphi\colon X \to \mathbb{R}$ be a real-valued function on X. Then φ is *eventually positive* if there exists $N \in \mathbb{N}$ such that $S_n\varphi(x) > 0$ for each $x \in X$ and each $n \in \mathbb{N}$ with $n \geqslant N$.

The following result is a consequence of the properties of the pressure function. For a proof, see [LZ24a, Corollary 3.29].

Lemma 2.12 (Li & Zheng [LZ24a])**.** *Let $f\colon S^2 \to S^2$ be an expanding Thurston map, and d be a visual metric on S^2 for f. Let $\phi \in C^{0,\beta}\big(S^2, d\big)$ be an eventually positive real-valued Hölder continuous function with an exponent $\beta \in (0, 1]$. Then the function $t \to P(f, -t\phi)$, $t \in \mathbb{R}$, is strictly decreasing and there exists a unique number $s_0 \in \mathbb{R}$ such that $P(f, -s_0\phi) = 0$. Moreover, $s_0 > 0$.*

We recall the strong non-integrability condition from [LZ24c, Subsection 7.1].

Definition 2.13 (Strong non-integrability condition)**.** Let $f\colon S^2 \to S^2$ be an expanding Thurston map and d be a visual metric on S^2 for f. Fix $\beta \in (0, 1]$. Let $\phi \in C^{0,\beta}\big(S^2, d\big)$ be a real-valued Hölder continuous function with an exponent β.

(1) We say that ϕ satisfies the $(\mathcal{C}, \beta)$-*strong non-integrability condition* (with respect to f and d), for a Jordan curve $\mathcal{C} \subseteq S^2$ with $\operatorname{post} f \subseteq \mathcal{C}$, if there exist
  (a) numbers N, $M \in \mathbb{N}$, $\varepsilon \in (0, 1)$,
  (b) M-tiles $Y^M_{\mathfrak{b}} \in \mathbf{X}^M_{\mathfrak{b}}(f, \mathcal{C})$, $Y^M_{\mathfrak{w}} \in \mathbf{X}^M_{\mathfrak{w}}(f, \mathcal{C})$
  such that for each $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, each integer $m \geqslant M$, and each m-tile $X \in \mathbf{X}^m(f, \mathcal{C})$ with $X \subseteq Y^M_{\mathfrak{c}}$, there exist two points x_1, $x_2 \in X$ with the following properties:
  (i) $\min\big\{d\big(x_1, S^2 \smallsetminus X\big), d\big(x_2, S^2 \smallsetminus X\big), d(x_1, x_2)\big\} \geqslant \operatorname{diam}_d(X)$, and
  (ii) for each integer $n \geqslant N$, there exist two $(n + M)$-tiles $X^{n+M}_{\mathfrak{c},1}$, $X^{n+M}_{\mathfrak{c},2} \in \mathbf{X}^{n+M}(f, \mathcal{C})$ such that $Y^M_{\mathfrak{c}} = f^n\big(X^{n+M}_{\mathfrak{c},1}\big) = f^n\big(X^{n+M}_{\mathfrak{c},2}\big)$ and
  $$|S_n\phi(\varsigma_1(x_1)) - S_n\phi(\varsigma_2(x_1)) - S_n\phi(\varsigma_1(x_2)) + S_n\phi(\varsigma_2(x_2))| \geqslant \varepsilon d(x_1, x_2)^\beta,$$
  where we write $\varsigma_i \coloneqq \big(f^n|_{X^{n+M}_{\mathfrak{c},i}}\big)^{-1}$ for each $i \in \{1, 2\}$.

(2) We say that ϕ satisfies the β-*strong non-integrability condition* (with respect to f and d) if ϕ satisfies the $(\mathcal{C}, \beta)$-strong non-integrability condition with respect to f and d for some Jordan curve $\mathcal{C} \subseteq S^2$ with $\operatorname{post} f \subseteq \mathcal{C}$.

(3) We say that ϕ satisfies the *strong non-integrability condition* (with respect to f and d) if ϕ satisfies the β'-strong non-integrability condition with respect to f and d for some $\beta' \in (0, \beta]$.

The strong non-integrability condition is independent of the choice of the Jordan curve $\mathcal{C}$ (see [LZ24c, Lemma 7.2]).

Definition 2.14 (Partition function)**.** Let $g\colon X \to X$ be a map on a topological space X. Let $\varphi\colon X \to \mathbb{C}$ be a complex-valued function on X. Consider $n \in \mathbb{N}$. We define a *partition function* $Z^{(n)}_{g,\varphi}(\cdot)\colon \mathbb{C} \to \mathbb{C}$ (for (g,φ)) as

$$Z^{(n)}_{g,\varphi}(s) := \sum_{x \in \operatorname{Fix}(g^n)} e^{sS_n\varphi(x)}, \quad s \in \mathbb{C}. \tag{2.14}$$

The primary tool employed in [Li18] for developing the thermodynamic formalism for expanding Thurston maps is the Ruelle operator. For our purposes in this paper, we require certain variants of the Ruelle operator, called *split Ruelle operators*, which were introduced in [LZ24c]. The relevant notions are recorded below.

Definition 2.15 (Partial split Ruelle operator)**.** Let $f\colon S^2 \to S^2$ be an expanding Thurston map, $\mathcal{C} \subseteq S^2$ a Jordan curve containing $\operatorname{post} f$, and $\psi \in C(S^2,\mathbb{C})$ a complex-valued continuous function. Let $n \in \mathbb{N}_0$, and $E \subseteq S^2$ a union of n-tiles in $\mathbf{X}^n(f,\mathcal{C})$. We define an operator $\mathcal{L}^{(n)}_{\psi,\mathfrak{c},E}\colon C(E,\mathbb{C}) \to C\big(X^0_{\mathfrak{c}},\mathbb{C}\big)$, for each $\mathfrak{c} \in \{\mathfrak{b},\mathfrak{w}\}$, by

$$\mathcal{L}^{(n)}_{\psi,\mathfrak{c},E}(u)(y) := \sum_{\substack{X^n \in \mathbf{X}^n_{\mathfrak{c}} \\ X^n \subseteq E}} u\big((f^n|_{X^n})^{-1}(y)\big) \exp\big(S_n\psi\big((f^n|_{X^n})^{-1}(y)\big)\big), \tag{2.15}$$

for $u \in C(E,\mathbb{C})$ and $y \in X^0_{\mathfrak{c}}$. When $E = X^0_{\mathfrak{s}}$ for some $\mathfrak{s} \in \{\mathfrak{b},\mathfrak{w}\}$, we often write

$$\mathcal{L}^{(n)}_{\psi,\mathfrak{c},\mathfrak{s}} := \mathcal{L}^{(n)}_{\psi,\mathfrak{c},X^0_{\mathfrak{s}}}.$$

For each $\mathfrak{c} \in \{\mathfrak{b},\mathfrak{w}\}$, we define the projection $\pi_{\mathfrak{c}}\colon C\big(X^0_{\mathfrak{b}},\mathbb{C}\big) \times C\big(X^0_{\mathfrak{w}},\mathbb{C}\big) \to C\big(X^0_{\mathfrak{c}},\mathbb{C}\big)$ by

$$\pi_{\mathfrak{c}}(u_{\mathfrak{b}},u_{\mathfrak{w}}) := u_{\mathfrak{c}}, \qquad \text{for } (u_{\mathfrak{b}},u_{\mathfrak{w}}) \in C\big(X^0_{\mathfrak{b}},\mathbb{C}\big) \times C\big(X^0_{\mathfrak{w}},\mathbb{C}\big).$$

Definition 2.16 (Split Ruelle operators)**.** Let $f\colon S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $f(\mathcal{C}) \subseteq \mathcal{C}$ and $\operatorname{post} f \subseteq \mathcal{C}$. Let d be a visual metric for f on S^2, and $\psi \in C^{0,\beta}((S^2,d),\mathbb{C})$ a complex-valued Hölder continuous function with an exponent $\beta \in (0,1]$. The *split Ruelle operator* $\mathbb{L}_\psi\colon C\big(X^0_{\mathfrak{b}}\big) \times C\big(X^0_{\mathfrak{w}}\big) \to C\big(X^0_{\mathfrak{b}}\big) \times C\big(X^0_{\mathfrak{w}}\big)$ is defined by

$$\mathbb{L}_\psi(u_{\mathfrak{b}},u_{\mathfrak{w}}) := \big(\mathcal{L}^{(1)}_{F,\varphi,\mathfrak{b},\mathfrak{b}}(u_{\mathfrak{b}}) + \mathcal{L}^{(1)}_{F,\varphi,\mathfrak{b},\mathfrak{w}}(u_{\mathfrak{w}}), \mathcal{L}^{(1)}_{F,\varphi,\mathfrak{w},\mathfrak{b}}(u_{\mathfrak{b}}) + \mathcal{L}^{(1)}_{F,\varphi,\mathfrak{w},\mathfrak{w}}(u_{\mathfrak{w}})\big) \tag{2.16}$$

for $u_{\mathfrak{b}} \in C\big(X^0_{\mathfrak{b}},\mathbb{C}\big)$ and $u_{\mathfrak{w}} \in C\big(X^0_{\mathfrak{w}},\mathbb{C}\big)$.

For all $n \in \mathbb{N}_0$, we write the operator norm

$$\left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \mathbb{L}^n_\psi \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert_\beta := \sup\Big\{ \big\|\pi_{\mathfrak{c}}\big(\mathbb{L}^n_\psi(u_{\mathfrak{b}},u_{\mathfrak{w}})\big)\big\|_{C^{0,\beta}} : \mathfrak{c} \in \{\mathfrak{b},\mathfrak{w}\},\ u_{\mathfrak{s}} \in C^{0,\beta}\big((X^0_{\mathfrak{s}},d),\mathbb{C}\big),\ \|u_{\mathfrak{s}}\|_{C^{0,\beta}} \leqslant 1,\ \mathfrak{s} \in \{\mathfrak{b},\mathfrak{w}\}\Big\}.$$

3. The Assumptions

We state below the hypotheses under which we will develop our theory in most parts of this paper. We will repeatedly refer to such assumptions in the later sections. We emphasize again that not all assumptions are assumed in all the statements in this paper.

The Assumptions.

(1) $f\colon S^2 \to S^2$ is an expanding Thurston map.
(2) $\mathcal{C} \subseteq S^2$ is a Jordan curve containing $\operatorname{post} f$ with the property that there exists an integer $n_{\mathcal{C}} \in \mathbb{N}$ such that $f^{n_{\mathcal{C}}}(\mathcal{C}) \subseteq \mathcal{C}$ and $f^m(\mathcal{C}) \nsubseteq \mathcal{C}$ for each $m \in \{1,2,\dots,n_{\mathcal{C}}-1\}$.
(3) d is a visual metric on S^2 for f with expansion factor $\Lambda > 1$.
(4) $\beta \in (0,1]$.
(5) $\phi \in C^{0,\beta}(S^2,d)$ is an eventually positive real-valued Hölder continuous function with exponent β.

(6) $s_0 \in \mathbb{R}$ is the unique positive real number satisfying $P(f, -s_0\phi) = 0$.
(7) $\alpha \coloneqq \frac{\mathrm{d}}{\mathrm{d}t} P(f, t\phi)|_{t=-s_0}$.
(8) μ_ϕ is the unique equilibrium state for the map f and the potential ϕ.

Note that the uniqueness of s_0 in (6) is guaranteed by Lemma 2.12. Furthermore, it follows from Theorem 2.9 (ii) and Definition 2.11 that

$$\alpha = \int \phi \,\mathrm{d}\mu_{-s_0\phi} > 0.$$

For a pair of f in (1) and ϕ in (5), we will say that a quantity depends on f and ϕ if it depends on s_0.

Observe that by Lemma 2.4, for each f satisfying (1), there exists at least one Jordan curve $\mathcal{C}$ satisfying (2). Since for a fixed f, the number $n_{\mathcal{C}}$ is uniquely determined by $\mathcal{C}$ in (2), in the remaining part of the paper we will say that a quantity depends on $\mathcal{C}$ even if it also depends on $n_{\mathcal{C}}$.

Recall that the expansion factor Λ of a visual metric d on S^2 for f is uniquely determined by d and f. We will say that a quantity depends on f and d if it depends on Λ.

In the discussion below, depending on the conditions we will need, we will sometimes say "Let f, $\mathcal{C}$, d, ϕ satisfy the Assumptions.", and sometimes say "Let f and $\mathcal{C}$ satisfy the Assumptions.", etc.

4. Pressure function and partition function estimates

In this section, we employ thermodynamic formalism and Ruelle operators to study the dynamics of expanding Thurston maps and derive decay estimates needed for the main theorem. We first investigate some differential and analytic properties of the topological pressure function in Subsection 4.1, then in Subsection 4.2 we establish several decay estimates associated with the Ruelle operators and partition functions.

4.1. **The topological pressure function.** We first introduce some terminology and then discuss properties of the topological pressure function.

The following proposition establishes the analyticity and second derivatives of the topological pressure function in suitable function directions by using the symbolic coding constructed in Proposition 2.6 and classical results from symbolic dynamics.

Proposition 4.1. *Let f and d satisfy the Assumptions in Section 3. Let ϕ, u, and v be Hölder continuous functions on S^2 with respect to d. Let μ_ϕ be the unique equilibrium state for f and ϕ. Then the following statements hold:*

(i) *The function $P(f, \phi + \cdot\, u)\colon \mathbb{R} \to \mathbb{R}$ is real-analytic. Moreover, there exists an open set $W \subseteq \mathbb{C}$ with $\mathbb{R} \subseteq W$ such that $P(f, \phi + \cdot\, u)$ extends to a complex-analytic function on W.*

(ii)

$$\frac{\partial^2}{\partial s \partial t} P(f, \phi + tu + sv)\Big|_{t=s=0} = \lim_{n\to+\infty} \frac{1}{n} \int \Big(S_n u - n \int u \,\mathrm{d}\mu_\phi \Big) \Big(S_n v - n \int v \,\mathrm{d}\mu_\phi \Big) \,\mathrm{d}\mu_\phi. \tag{4.1}$$

In particular,

$$\sigma^2_{\mu_\phi}(u) \coloneqq \frac{\mathrm{d}^2}{\mathrm{d}t^2} P(f, \phi + tu)\Big|_{t=0} = \lim_{n\to+\infty} \frac{1}{n} \int \Big(S_n u - n \int u \,\mathrm{d}\mu_\phi \Big)^2 \,\mathrm{d}\mu_\phi.$$

Moreover, $\sigma^2_{\mu_\phi}(u) = 0$ if and only if there exists $w \in C(S^2)$ such that $u = \int u \,\mathrm{d}\mu_\phi + w \circ f - w$.

Proof. We reduce the general case to one where the map possesses an invariant Jordan curve, thereby permitting the use of symbolic dynamics. By Lemma 2.4, we may take a sufficiently high iterate $F \coloneqq f^K$ that admits an F-invariant Jordan curve $\mathcal{C} \subseteq S^2$ containing the postcritical set $\operatorname{post} F = \operatorname{post} f$. The map F is then an expanding Thurston map for which d is a visual metric (cf. [BM17, Proposition 8.3 (v)]). We fix this integer K, map F, and curve $\mathcal{C}$.

Applying Proposition 2.6 to F and $\mathcal{C}$, we denote the corresponding one-sided subshift of finite type by (Σ^+, σ) and the factor map by $\pi\colon \Sigma^+ \to S^2$. Fix $\tau \in (0,1)$ and equip the space Σ^+ with the metric d_τ defined in (2.6). Then $\pi \circ \sigma = F \circ \pi$, and $P(F, \psi) = P(\sigma, \psi \circ \pi)$ for each Hölder continuous function $\psi\colon S^2 \to \mathbb{R}$ (with respect to d). Here the lifted potential $\psi \circ \pi\colon \Sigma^+ \to \mathbb{R}$ is Hölder continuous with respect to the metric d_τ on Σ^+.

Denote $\Phi \coloneqq S_K^f \phi$, $U \coloneqq S_K^f u$, and $V \coloneqq S_K^f v$. Then Φ, U, and V are Hölder continuous with respect to d since f is Lipschitz continuous with respect to d (cf. [Li17, Lemma 3.12]).

Let $\mu_{F,\Phi}$ be the unique equilibrium state for F and Φ. Since $P(F, \Phi) = KP(f, \phi)$ (cf. [Wal82, Theorem 9.8 (i)]), it follows from $P_{\mu_\phi}(F, \Phi) = KP_{\mu_\phi}(f, \phi)$ and the uniqueness of the equilibrium state that $\mu_{F,\Phi} = \mu_\phi$.

(i) By [PP90, Proposition 4.7], the function $P(\sigma, \Phi \circ \pi + \cdot\, U \circ \pi)\colon \mathbb{R} \to \mathbb{R}$ is real-analytic. Since $P(F, \Phi + tU) = P(\sigma, \Phi \circ \pi + tU \circ \pi)$ and $P(F, \Phi + tU) = P\big(f^K, S_K^f(\phi + tu)\big) = KP(f, \phi + tu)$ for all $t \in \mathbb{R}$, we conclude that $t \mapsto P(f, \phi + tu)$ is real-analytic, and thus extends to a complex-analytic function on a neighborhood of $\mathbb{R}$.

(ii) We first establish the formula (4.1). By [Bow75, Theorem 1.22], there exists a unique equilibrium state $\nu_{\Phi\circ\pi} \in \mathcal{M}(\Sigma^+, \sigma)$ for σ and $\Phi \circ \pi$. Then it follows from [PU10, Theorem 5.7.4] that

$$
\begin{aligned}
&\frac{\partial^2}{\partial s \partial t} P(\sigma, \Phi \circ \pi + tU \circ \pi + sV \circ \pi)\Big|_{t=s=0} \\
&\quad = \lim_{n \to +\infty} \frac{1}{n} \int \bigg(S_n^\sigma(U \circ \pi) - n \int U \circ \pi \,\mathrm{d}\nu_{\Phi\circ\pi} \bigg) \bigg(S_n^\sigma(V \circ \pi) - n \int V \circ \pi \,\mathrm{d}\nu_{\Phi\circ\pi} \bigg) \,\mathrm{d}\nu_{\Phi\circ\pi}.
\end{aligned}
$$

Denote $\mu \coloneqq \pi_* \nu_{\Phi\circ\pi} \in \mathcal{M}(S^2, F)$. Then by Lemma 2.10, we have $\mu = \mu_{F,\Phi} = \mu_\phi$. Since $P(f, \phi + tu + sv) = \frac{1}{K} P(F, \Phi + tU + sV)$ and $\int \varphi \,\mathrm{d}\mu = \int \varphi \circ \pi \,\mathrm{d}\nu_{\phi\circ\pi}$ for all $\varphi \in C(S^2)$, we get

$$
\frac{\partial^2}{\partial s \partial t} P(f, \phi + tu + sv)\Big|_{t=s=0} = \frac{1}{K} \lim_{n \to +\infty} \frac{1}{n} \int \big(S_n^F \widetilde{U}\big)\big(S_n^F \widetilde{V}\big) \,\mathrm{d}\mu_\phi = \lim_{n \to +\infty} \frac{1}{nK} \int \big(S_{nK}^f \widetilde{u}\big)\big(S_{nK}^f \widetilde{v}\big) \,\mathrm{d}\mu_\phi,
$$

where $\widetilde{U} \coloneqq U - \int U \,\mathrm{d}\mu_\phi$ and $\widetilde{u} \coloneqq u - \int u \,\mathrm{d}\mu_\phi$ (and similarly for $\widetilde{V}$ and $\widetilde{v}$). To establish (4.1), it suffices to verify that $\lim_{m \to +\infty} \frac{1}{m} \int \big(S_m^f \widetilde{u}\big)\big(S_m^f \widetilde{v}\big) \,\mathrm{d}\mu_\phi$ exists.

For each $m \in \mathbb{N}$, we write $m = nK + r$ with $r \in \{0, \dots, K-1\}$. We write $S_m^f \widetilde{u} = S_{nK}^f \widetilde{u} + E_m(\widetilde{u})$, where the error term $E_m(\widetilde{u}) \coloneqq S_r^f(\widetilde{u} \circ f^{nK})$ satisfies $\|E_m(\widetilde{u})\|_\infty \leqslant K \|\widetilde{u}\|_\infty$. Then by the Cauchy–Schwarz inequality, we obtain

$$
\begin{aligned}
&\bigg| \frac{1}{m} \int \big(S_m^f \widetilde{u}\big)\big(S_m^f \widetilde{v}\big) \,\mathrm{d}\mu_\phi - \frac{1}{m} \int \big(S_{nK}^f \widetilde{u}\big)\big(S_{nK}^f \widetilde{v}\big) \,\mathrm{d}\mu_\phi \bigg| \\
&\quad = \frac{1}{m} \bigg| \int E_m(\widetilde{v}) S_{nK}^f \widetilde{u} \,\mathrm{d}\mu_\phi + \int E_m(\widetilde{u}) S_{nK}^f \widetilde{v} \,\mathrm{d}\mu_\phi + \int E_m(\widetilde{u}) E_m(\widetilde{v}) \,\mathrm{d}\mu_\phi \bigg| \\
&\quad \leqslant \frac{1}{m} \Big(K \|\widetilde{v}\|_\infty \big\| S_{nK}^f \widetilde{u} \big\|_{L^2(\mu_\phi)} + K \|\widetilde{u}\|_\infty \big\| S_{nK}^f \widetilde{v} \big\|_{L^2(\mu_\phi)} + K^2 \|\widetilde{u}\|_\infty \|\widetilde{v}\|_\infty \Big).
\end{aligned}
$$

Since $\lim_{n \to +\infty} \frac{1}{nK} \big\| S_{nK}^f \widetilde{u} \big\|_{L^2(\mu_\phi)}^2 = \frac{\mathrm{d}^2}{\mathrm{d}t^2} P(f, \phi + tu)\Big|_{t=0}$, we have $\big\| S_{nK}^f \widetilde{u} \big\|_{L^2(\mu_\phi)} = O(\sqrt{n})$ as $n \to \infty$ (and similarly for $\widetilde{v}$). Since $\lim_{m \to \infty} m/(nK) = 1$, we conclude that

$$
\lim_{m \to +\infty} \frac{1}{m} \int \big(S_m^f \widetilde{u}\big)\big(S_m^f \widetilde{v}\big) \,\mathrm{d}\mu_\phi = \lim_{n \to +\infty} \frac{1}{nK} \int \big(S_{nK}^f \widetilde{u}\big)\big(S_{nK}^f \widetilde{v}\big) \,\mathrm{d}\mu_\phi.
$$

Finally, the last statement follows from [DPTUZ21, Theorem 1.2 (6)]. $\square$

Theorem 4.1 implies the analyticity and expansion of the pressure function with respect to the imaginary part of the potential.

Lemma 4.2. *Let f, d, ϕ, s_0, α satisfy the Assumptions in Section 3. Let $\mu_{-s_0\phi}$ be the unique equilibrium state for the map f and the potential $-s_0\phi$. We assume that ϕ is not cohomologous to a* *constant in $C(S^2)$. Denote $\overline{\phi} \coloneqq \phi - \alpha$. Then there exist $\delta > 0$ and $C_\delta \geqslant 0$ such that the function $t \mapsto P(f, (-s_0 + \mathbf{i}t)\overline{\phi}) \colon (-\delta, \delta) \to \mathbb{C}$ is complex-analytic, and for each $t \in (-\delta, \delta)$,*

$$\left| P\big(f, (-s_0 + \mathbf{i}t)\overline{\phi}\big) - P(f, -s_0\overline{\phi}) + \frac{1}{2}\sigma^2 t^2 \right| \leqslant C_\delta |t|^3, \tag{4.2}$$

where $\sigma > 0$ is given by

$$\sigma^2 \coloneqq \frac{\mathrm{d}^2}{\mathrm{d}t^2} P(f, t\phi)\big|_{t=-s_0} = \lim_{n\to+\infty} \frac{1}{n} \int (S_n\phi - n\alpha)^2 \,\mathrm{d}\mu_{-s_0\phi}. \tag{4.3}$$

Proof. By Proposition 4.1 (i), there exists $\delta_1 > 0$ with $-s_0 \in B(-s_0, \delta_1) \subseteq \mathbb{C}$ such that $P(f, \cdot\,\phi) \colon B(-s_0, \delta_1) \to \mathbb{C}$ is complex-analytic. Recall that $\alpha = \frac{\mathrm{d}}{\mathrm{d}t} P(f, t\phi)|_{t=-s_0}$. Then it follows from Taylor's formula and (4.3) that there exist constants $\delta \in (0, \delta_1]$ and $C_\delta \geqslant 0$ such that for each $t \in (-\delta, \delta)$,

$$\left| P(f, (-s_0 + \mathbf{i}t)\phi) - \Big(P(f, -s_0\phi) + \mathbf{i}\alpha t - \frac{1}{2}\sigma^2 t^2 \Big) \right| \leqslant C_\delta |t|^3.$$

This implies (4.2) since $P(f, (-s_0 + \mathbf{i}t)\phi) = P\big(f, (-s_0 + \mathbf{i}t)\overline{\phi}\big) + (-s_0 + \mathbf{i}t)\alpha$ and $P(f, -s_0\phi) = P(f, -s_0\overline{\phi}) - s_0\alpha$. ☐

4.2. **Decay estimates of the partition function.** In this subsection, we investigate partition functions and obtain several decay estimates to prove the main theorem. Our approach is inspired by the recent work [LZ24a, LZ24c, LZ24b] on prime orbit theorems for expanding Thurston maps.

The following result proved in [LZ24c, Proposition 6.1] is a version of Ruelle's estimate adapted to our setting. The idea of the proof originated from Ruelle [Rue90].

Proposition 4.3 (Z. Li & T. Zheng [LZ24c])**.** *Let f, $\mathcal{C}$, d, Λ, β, ϕ, s_0 satisfy the Assumptions in Section 3. We assume that $f(\mathcal{C}) \subseteq \mathcal{C}$ and no 1-tile in $\mathbf{X}^1(f, \mathcal{C})$ joins opposite sides of $\mathcal{C}$. Let $\big(\Sigma_{A_\vartriangle}^+, \sigma_{A_\vartriangle}\big)$ be the one-sided subshift of finite type associated to f and $\mathcal{C}$ defined in Proposition 2.6, and let $\pi_\vartriangle \colon \Sigma_{A_\vartriangle}^+ \to S^2$ be defined in (2.8).*

Then for each $\delta > 0$ there exists a constant $D_\delta > 0$ such that for all integers $n \geqslant 2$ and $k \in \mathbb{N}$, we have

$$\sum_{X^k \in \mathbf{X}^k(f,\mathcal{C})} \max_{\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}} \big\| \mathcal{L}^{(k)}_{-s\phi, \mathfrak{c}, X^k}(\mathbb{1}_{X^k}) \big\|_{C^{0,\beta}} \leqslant D_\delta |\mathrm{Im}(s)| \Lambda^{-\beta} \exp(k(\delta + P(f, -\mathrm{Re}(s)\phi))). \tag{4.4}$$

and

$$\begin{aligned} &\Bigg| Z^{(n)}_{\sigma_{A_\vartriangle}, -\phi\circ\pi_\vartriangle}(s) - \sum_{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}} \sum_{\substack{X \in \mathbf{X}^1(f,\mathcal{C}) \\ X \subseteq X^0_{\mathfrak{c}}}} \mathcal{L}^{(n)}_{-s\phi, \mathfrak{c}, X}(\mathbb{1}_X)(x_X) \Bigg| \\ &\qquad \leqslant D_\delta |\mathrm{Im}(s)| \sum_{m=2}^{n} \big|\!\big|\!\big| \mathbb{L}^{n-m}_{-s\phi} \big|\!\big|\!\big|_\beta \big(\Lambda^{-\beta} \exp(\delta + P(f, -\mathrm{Re}(s)\phi))\big)^m \end{aligned} \tag{4.5}$$

for any choice of a point $x_X \in \mathrm{inte}(X)$ for each $X \in \mathbf{X}^1(f, \mathcal{C})$, and for all $s \in \mathbb{C}$ with $|\mathrm{Im}(s)| \geqslant 2s_0 + 1$ and $|\mathrm{Re}(s) - s_0| \leqslant s_0$, where $Z^{(n)}_{\sigma_{A_\vartriangle}, -\phi\circ\pi_\vartriangle}(s)$ is defined in Definition 2.14.

We record [LZ24c, Theorem 6.3] here, which establishes an exponential bound on the operator norm of the split Ruelle operator. We remark that by [LZ24c, Proposition 7.3] and [LZ24a, Theorem F], if ϕ satisfies the strong non-integrability condition (Definition 2.13), then ϕ is not cohomologous to a constant in $C(S^2)$.

Theorem 4.4 (Z. Li & T. Zheng [LZ24c])**.** *Let f, $\mathcal{C}$, d, Λ, β, ϕ, s_0 satisfy the Assumptions in Section 3. We assume that $f(\mathcal{C}) \subseteq \mathcal{C}$ and ϕ satisfies the β-strong non-integrability condition. Then there exists a constant $D' = D'(f, \mathcal{C}, d, \beta, \phi) > 0$ such that for each $\varepsilon > 0$ there exist constants $\delta_\varepsilon \in (0, s_0)$, $\widetilde{b}_\varepsilon \geqslant 2s_0 + 1$, and $\rho_\varepsilon \in (0, 1)$ with the following property:*

For each $n \in \mathbb{N}$ and each $s \in \mathbb{C}$ satisfying $|\mathrm{Re}(s) - s_0| < \delta_\varepsilon$ and $|\mathrm{Im}(s)| \geqslant \widetilde{b}_\varepsilon$, we have

$$\left\| \mathbb{L}^n_{-s\phi} \right\|_\beta \leqslant D' |\mathrm{Im}(s)|^{1+\varepsilon} \rho^n_\varepsilon. \tag{4.6}$$

We first study the partition function for the symbolic coding of an expanding Thurston map, separating the estimates into unbounded and bounded cases based on the imaginary part of s.

In the unbounded case, the following estimate is a direct consequence of Proposition 4.3 and Theorem 4.4. We adapt the strategy of [LZ24c, Proposition 6.4]; we include the proof for the sake of completeness.

Proposition 4.5. *Let f, $\mathcal{C}$, d, Λ, β, ϕ, s_0 satisfy the Assumptions in Section 3. We assume that ϕ satisfies the β-strong non-integrability condition, and that $f(\mathcal{C}) \subseteq \mathcal{C}$ and no 1-tile in $\mathbf{X}^1(f,\mathcal{C})$ joins opposite sides of $\mathcal{C}$. Let $\left(\Sigma^+_{A_\triangle}, \sigma_{A_\triangle}\right)$ be the one-sided subshift of finite type associated to f and $\mathcal{C}$ defined in Proposition 2.6, and let $\pi_\triangle \colon \Sigma^+_{A_\triangle} \to S^2$ be defined in (2.8).*

Then for each $\varepsilon > 0$, there exist constants $T > 1$, $C_\triangle > 0$, and $\rho_\triangle \in (0,1)$ such that for each $t \in \mathbb{R} \smallsetminus (-T,T)$ and each integer $n \geqslant 2$, we have

$$\left| Z^{(n)}_{\sigma_{A_\triangle},\, \phi\circ\pi_\triangle}(-s_0 + \mathbf{i}t) \right| \leqslant C_\triangle |t|^{2+\varepsilon} \rho^n_\triangle. \tag{4.7}$$

Proof. Let $\delta := \frac{1}{2}\log(\Lambda^\beta) > 0$. Fix an arbitrary $\varepsilon > 0$. We choose $T := \widetilde{b}_\varepsilon > 1$, where $\widetilde{b}_\varepsilon$ is the constant from Theorem 4.4 depending only on f, $\mathcal{C}$, d, β, ϕ, and ε.

Fix an arbitrary point $x_{X^1} \in \operatorname{inte}(X^1)$ for each $X^1 \in \mathbf{X}^1$. Recall the definition of partial split Ruelle operators in Definition 2.15 (see also [LZ24c, Lemmas 5.3 and 5.7]). By applying the estimate (4.4) from Proposition 4.3, for each $n \geqslant 2$ and each $t \in \mathbb{R} \smallsetminus (-T,T)$, we obtain the following bound:

$$\begin{aligned}
I &:= \left| \sum_{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}} \sum_{\substack{X^1\in\mathbf{X}^1 \\ X^1\subseteq X^0_\mathfrak{c}}} \mathcal{L}^{(n)}_{(-s_0+\mathbf{i}t)\phi,\mathfrak{c},X^1}(\mathbb{1}_{X^1})(x_{X^1}) \right| \\
&\leqslant \sum_{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}} \sum_{\substack{X^1\in\mathbf{X}^1 \\ X^1\subseteq X^0_\mathfrak{c}}} \left| \sum_{\mathfrak{c}'\in\{\mathfrak{b},\mathfrak{w}\}} \mathcal{L}^{(n-1)}_{(-s_0+\mathbf{i}t)\phi,\mathfrak{c},\mathfrak{c}'} \Big(\mathcal{L}^{(1)}_{(-s_0+\mathbf{i}t)\phi,\mathfrak{c}',X^1}(\mathbb{1}_{X^1}) \Big)(x_{X^1}) \right| \\
&\leqslant \left\| \mathbb{L}^{n-1}_{(-s_0+\mathbf{i}t)\phi} \right\|_\beta \sum_{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}} \sum_{\substack{X^1\in\mathbf{X}^1 \\ X^1\subseteq X^0_\mathfrak{c}}} \max_{\mathfrak{c}'\in\{\mathfrak{b},\mathfrak{w}\}} \left\| \mathcal{L}^{(1)}_{(-s_0+\mathbf{i}t)\phi,\mathfrak{c}',X^1}(\mathbb{1}_{X^1}) \right\|_{C^{0,\beta}} \\
&\leqslant \left\| \mathbb{L}^{n-1}_{(-s_0+\mathbf{i}t)\phi} \right\|_\beta D_\delta |t| \Lambda^{-\beta} \exp(\delta + P(f,-s_0\phi)) \\
&= \left\| \mathbb{L}^{n-1}_{(-s_0+\mathbf{i}t)\phi} \right\|_\beta D_\delta |t| \Lambda^{-\beta/2},
\end{aligned} \tag{4.8}$$

where $D_\delta > 0$ is the constant given in Proposition 4.3, which depends only on f, $\mathcal{C}$, d, β, ϕ, and δ.

Hence, by (2.14), the triangle inequality, (4.5) in Proposition 4.3, (4.8), and Theorem 4.4, we deduce that for each $n \geqslant 2$ and each $t \in \mathbb{R} \smallsetminus (-T,T)$,

$$\begin{aligned}
\left| Z^{(n)}_{\sigma_{A_\triangle},\, \phi\circ\pi_\triangle}(-s_0 + \mathbf{i}t) \right| &\leqslant I + \left| Z^{(n)}_{\sigma_{A_\triangle},\, \phi\circ\pi_\triangle}(-s_0 + \mathbf{i}t) - \sum_{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}} \sum_{\substack{X^1\in\mathbf{X}^1 \\ X^1\subseteq X^0_\mathfrak{c}}} \mathcal{L}^{(n)}_{(-s_0+\mathbf{i}t)\phi,\mathfrak{c},X^1}(\mathbb{1}_{X^1})(x_{X^1}) \right| \\
&\leqslant D_\delta |t| \left\| \mathbb{L}^{n-1}_{(-s_0+\mathbf{i}t)\phi} \right\|_\beta \Lambda^{-\beta/2} + D_\delta |t| \sum_{m=2}^{n} \left\| \mathbb{L}^{n-m}_{(-s_0+\mathbf{i}t)\phi} \right\|_\beta \Lambda^{-\beta m/2} \\
&\leqslant D_\delta D' |t|^{2+\varepsilon} \sum_{m=1}^{n} \rho^{n-m}_\varepsilon \Lambda^{-\beta m/2} \\
&\leqslant D_\delta D' |t|^{2+\varepsilon}\, n \big(\max\{\rho_\varepsilon, \Lambda^{-\beta/2}\} \big)^n,
\end{aligned}$$

where constants $D' > 0$ and $\rho_\varepsilon \in (0,1)$ are from Theorem 4.4 depending only on f, $\mathcal{C}$, d, β, ϕ, and ε.

Therefore, we may choose constants

$$\rho_\triangle := \frac{1}{2}\big(1 + \max\big\{\rho_\varepsilon, \Lambda^{-\beta/2}\big\}\big) \in (0,1)$$

and

$$C_\triangle := D_\delta D' \sup_{n\in\mathbb{N}}\{n\rho_\triangle^n\} < +\infty$$

to ensure that the estimate (4.7) holds. □

We record the following result from [LZ24a, Theorem 6.8].

Theorem 4.6 (Li & Zheng [LZ24a]). *Let f, $\mathcal{C}$, d satisfy the Assumptions in Section 3. We assume that $f(\mathcal{C}) \subseteq \mathcal{C}$. Let $\varphi \in C^{0,\beta}(S^2,d)$ be a real-valued Hölder continuous function with exponent β. Recall the one-sided subshifts of finite type $\big(\Sigma_{A_{\shortmid}}^+, \sigma_{A_{\shortmid}}\big)$ and $\big(\Sigma_{A_{\shortparallel}}^+, \sigma_{A_{\shortparallel}}\big)$ constructed in Subsection 2.4. We denote by $(\mathbf{V}^0, f|_{\mathbf{V}^0})$ the dynamical system on $\mathbf{V}^0 := \mathbf{V}^0(f,\mathcal{C}) = \operatorname{post} f$ induced by $f|_{\mathbf{V}^0} \colon \mathbf{V}^0 \to \mathbf{V}^0$. Then the following relations between the topological pressure of these systems hold:*

(i) $P(f,\varphi) > P(f|_{\mathbf{V}^0}, \varphi|_{\mathbf{V}^0})$,

(ii) $P(f,\varphi) > P(f|_{\mathcal{C}}, \varphi|_{\mathcal{C}}) = P(\sigma_{A_{\shortmid}}, \varphi \circ \pi_{\shortmid}) = P(\sigma_{A_{\shortparallel}}, \varphi\circ\pi_{\shortmid}\circ\pi_{\shortparallel})$.

We record the following result from [LZ24a, Theorem 6.3].

Theorem 4.7 (Li & Zheng [LZ24a]). *Let f and $\mathcal{C}$ satisfy the Assumptions in Section 3. We assume that $f(\mathcal{C}) \subseteq \mathcal{C}$. Let $\big(\Sigma_{A_\triangle}^+, \sigma_{A_\triangle}\big)$ be the one-sided subshift of finite type associated to f and $\mathcal{C}$ defined in Proposition 2.6, and let $\pi_\triangle \colon \Sigma_{A_\triangle}^+ \to S^2$ be the factor map defined in (2.8). Recall the one-sided subshifts of finite type $\big(\Sigma_{A_{\shortmid}}^+, \sigma_{A_{\shortmid}}\big)$ and $\big(\Sigma_{A_{\shortparallel}}^+, \sigma_{A_{\shortparallel}}\big)$ constructed in Subsection 2.4, and the factor maps $\pi_{\shortmid} \colon \Sigma_{A_{\shortmid}}^+ \to \mathcal{C}$ and $\pi_{\shortparallel} \colon \Sigma_{A_{\shortparallel}}^+ \to \Sigma_{A_{\shortmid}}^+$ defined in Proposition 2.8. We denote by $(\mathbf{V}^0, f|_{\mathbf{V}^0})$ the dynamical system on $\mathbf{V}^0 = \mathbf{V}^0(f,\mathcal{C}) = \operatorname{post} f$ induced by $f|_{\mathbf{V}^0} \colon \mathbf{V}^0 \to \mathbf{V}^0$. Then for each $n \in \mathbb{N}$ and each $x \in \operatorname{Fix}(f^n)$, we have*

$$\deg_{f^n}(x) = M_{A_\triangle}(x,n) - M_{A_{\shortparallel}}(x,n) + M_{A_{\shortmid}}(x,n) + M_\bullet(x,n), \tag{4.9}$$

where

$$\begin{aligned} M_{A_\triangle}(x,n) &:= \operatorname{card}\big(\operatorname{Fix}\big(\sigma_{A_\triangle}^n\big) \cap \pi_\triangle^{-1}(x)\big), \\ M_{A_{\shortparallel}}(x,n) &:= \operatorname{card}\big(\operatorname{Fix}\big(\sigma_{A_{\shortparallel}}^n\big) \cap (\pi_{\shortmid}\circ\pi_{\shortparallel})^{-1}(x)\big), \\ M_{A_{\shortmid}}(x,n) &:= \operatorname{card}\big(\operatorname{Fix}\big(\sigma_{A_{\shortmid}}^n\big) \cap \pi_{\shortmid}^{-1}(x)\big), \\ M_\bullet(x,n) &:= \operatorname{card}(\operatorname{Fix}((f|_{\mathbf{V}^0})^n) \cap \{x\}). \end{aligned}$$

By definition, $M_{A_{\shortparallel}}(x,n) = M_{A_{\shortmid}}(x,n) = 0$ for all $x \notin \mathcal{C}$, and $M_\bullet(x,n) = 0$ for all $x \notin \mathbf{V}^0$.

Lemma 4.8. *Let X and Y be topological spaces, and let $G\colon Y \to Y$ and $g\colon X \to X$ be continuous maps. If (X,g) is a factor of (Y,G) with a factor map $\pi\colon Y \to X$, then for each $n \in \mathbb{N}$, we have $\pi(\operatorname{Fix}(G^n)) \subseteq \operatorname{Fix}(g^n)$, and the set $\operatorname{Fix}(G^n)$ of periodic points admits the decomposition*

$$\operatorname{Fix}(G^n) = \bigcup_{x\in\operatorname{Fix}(g^n)} \big(\operatorname{Fix}(G^n) \cap \pi^{-1}(x)\big).$$

Moreover, for each complex-valued function $\varphi\colon X \to \mathbb{C}$, we have

$$\sum_{y\in\operatorname{Fix}(G^n)} e^{S_n\varphi(\pi(y))} = \sum_{x\in\operatorname{Fix}(g^n)} \operatorname{card}\big(\operatorname{Fix}\big(G^n\big) \cap \pi^{-1}(x)\big)\, e^{S_n\varphi(x)}$$

Proof. For each $y \in \operatorname{Fix}(G^n)$ we have $g^n(\pi(y)) = \pi(G^n(y)) = \pi(y)$ since $g \circ \pi = \pi \circ G$. Thus $\pi(y) \in \operatorname{Fix}(g^n)$, which establishes the inclusion $\pi(\operatorname{Fix}(G^n)) \subseteq \operatorname{Fix}(g^n)$.

The decomposition of $\operatorname{Fix}(G^n)$ follows directly. The inclusion $\supseteq$ is trivial. For the reverse inclusion, every $y \in \operatorname{Fix}(G^n)$ belongs to $\operatorname{Fix}(G^n) \cap \pi^{-1}(\pi(y))$. Since $\pi(y) \in \operatorname{Fix}(g^n)$, this set is part of the union.

The identity for the sum over periodic points is obtained by grouping terms according to the decomposition:

$$\sum_{y \in \operatorname{Fix}(G^n)} e^{S_n \varphi(\pi(y))} = \sum_{x \in \operatorname{Fix}(g^n)} \sum_{y \in \operatorname{Fix}(G^n) \cap \pi^{-1}(x)} e^{S_n \varphi(\pi(y))}.$$

Since $\pi(y) = x$ for every y in the inner sum, this simplifies to

$$\sum_{x \in \operatorname{Fix}(g^n)} \sum_{y \in \operatorname{Fix}(G^n) \cap \pi^{-1}(x)} e^{S_n \varphi(x)} = \sum_{x \in \operatorname{Fix}(g^n)} \operatorname{card}\bigl(\operatorname{Fix}(G^n) \cap \pi^{-1}(x)\bigr)\, e^{S_n \varphi(x)}.$$

This completes the proof. ☐

Lemma 4.9. *Consider a finite set of states S and a transition matrix $A \colon S \times S \to \{0,1\}$. Let $\bigl(\Sigma_A^+, \sigma_A\bigr)$ be the one-sided subshift of finite type defined by A, and $\psi \in C\bigl(\Sigma_A^+\bigr)$ be a real-valued continuous function. Then for each $\varepsilon > 0$ there exists $C = C(\varepsilon) > 0$ such that for all $n \in \mathbb{N}$,*

$$\sum_{y \in \operatorname{Fix}(\sigma_A^n)} e^{S_n \psi(y)} \leqslant C e^{n(P(\sigma_A, \psi) + \varepsilon)}. \tag{4.10}$$

Proof. Let $n \in \mathbb{N}$ and $Z_n(\psi) \coloneqq \sum_{y \in \operatorname{Fix}(\sigma_A^n)} e^{S_n \psi(y)}$. Any two distinct points in $\operatorname{Fix}(\sigma_A^n)$ must differ in at least one of their first n coordinates, which implies that $\operatorname{Fix}(\sigma_A^n)$ is an $(n,1)$-separated set (see Subsection 2.2). Then it follows directly from the definition of topological pressure (2.1) that

$$\limsup_{n \to +\infty} \frac{1}{n} \log Z_n(\psi) \leqslant P(\sigma_A, \psi).$$

Given $\varepsilon > 0$, this implies $Z_n(\psi) \leqslant e^{n(P(\sigma_A, \psi) + \varepsilon)}$ for all sufficiently large n. The claimed inequality (4.10) for all $n \in \mathbb{N}$ then follows by choosing a sufficiently large constant $C = C(\varepsilon) > 0$. ☐

Lemma 4.10. *Let f, $\mathcal{C}$, d, ϕ, s_0 satisfy the Assumptions in Section 3. We assume that $f(\mathcal{C}) \subseteq \mathcal{C}$ and no 1-tile in $\mathbf{X}^1(f, \mathcal{C})$ joins opposite sides of $\mathcal{C}$. Then there exist constants $\kappa \in (0,1)$ and $C > 0$ such that for each $n \in \mathbb{N}$ and each $s \in \mathbb{C}$ with $\operatorname{Re}(s) = -s_0$, we have*

$$\bigl| Z_{f,\,\phi}^{(n)}(s) - Z_{\sigma_{A_\Delta},\,\phi \circ \pi_\Delta}^{(n)}(s) \bigr| \leqslant C \kappa^n.$$

Proof. Let $s \in \mathbb{C}$ with $\operatorname{Re}(s) = -s_0$.

We begin by decomposing the partition function $Z_{f,\,\phi}^{(n)}(s)$ into two terms:

$$Z_{f,\,\phi}^{(n)}(s) = \sum_{x \in \operatorname{Fix}(f^n)} \deg_{f^n}(x) e^{s S_n \phi(x)} - \sum_{x \in \operatorname{Fix}(f^n)} \bigl(\deg_{f^n}(x) - 1\bigr) e^{s S_n \phi(x)}. \tag{4.11}$$

Substituting the expression (4.9) in Theorem 4.7 for the local degree $\deg_{f^n}(x)$ transforms the first sum on the right-hand side of (4.11) into

$$\sum_{x \in \operatorname{Fix}(f^n)} \bigl(M_{A_\Delta}(x,n) - M_{A_{\shortparallel}}(x,n) + M_{A_{\shortmid}}(x,n) + M_\bullet(x,n) \bigr) e^{s S_n \phi(x)}.$$

By Lemma 4.8, this sum can be expressed in terms of the associated symbolic systems via the factor maps π_Δ, $\pi_{\shortmid} \circ \pi_{\shortparallel}$, and $\pi_{\shortmid}$ (cf. Propositions 2.6 and 2.8), yielding

$$Z_{\sigma_{A_\Delta},\,\phi \circ \pi_\Delta}^{(n)}(s) - \sum_{y \in \operatorname{Fix}(\sigma_{A_{\shortparallel}}^n)} e^{s S_n \phi(\pi_{\shortmid} \circ \pi_{\shortparallel}(y))} + \sum_{y \in \operatorname{Fix}(\sigma_{A_{\shortmid}}^n)} e^{s S_n \phi(\pi_{\shortmid}(y))} + \sum_{x \in \operatorname{Fix}((f|_{\mathbf{V}^0})^n)} e^{s S_n \phi(x)}.$$

Collecting all terms, we obtain the decomposition

$$Z^{(n)}_{f,\phi}(s) = Z^{(n)}_{\sigma_{A_\Delta},\,\phi\circ\pi_\Delta}(s) - \mathrm{I}_n(s) - \Pi_n(s),$$

where

$$\mathrm{I}_n(s) \coloneqq \sum_{x\in\operatorname{Fix}(f^n)} \bigl(\deg_{f^n}(x) - 1\bigr) e^{sS_n\phi(x)}$$

and

$$\Pi_n(s) \coloneqq \sum_{y\in\operatorname{Fix}(\sigma^n_{A_{\shortparallel}})} e^{sS_n\phi\circ\pi_{\shortmid}\circ\pi_{\shortparallel}(y)} - \sum_{y\in\operatorname{Fix}(\sigma^n_{A_{\shortmid}})} e^{sS_n\phi\circ\pi_{\shortmid}(y)} - \sum_{x\in\operatorname{Fix}((f|_{\mathbf{V}^0})^n)} e^{sS_n\phi(x)}.$$

It remains to show that $|\mathrm{I}_n(s)|$ and $|\Pi_n(s)|$ are bounded by terms that decay exponentially in n.

We first estimate $|\mathrm{I}_n(s)|$. The only non-zero terms in the sum defining $\mathrm{I}_n(s)$ correspond to periodic points x with $\deg_{f^n}(x) > 1$, which are necessarily contained in the postcritical set $\mathbf{V}^0 = \operatorname{post} f$. To analyze these terms, we introduce some notation. For a primitive periodic orbit $\tau \in \mathcal{P}(f)$, we write

$$l_{f,\phi}(\tau) \coloneqq \sum_{y\in\tau} \phi(y) \quad\text{and}\quad \deg_f(\tau) \coloneqq \prod_{y\in\tau} \deg_f(y).$$

Define $\mathcal{P}^{>}(f|_{\mathbf{V}^0}) \coloneqq \{\tau \in \mathcal{P}(f|_{\mathbf{V}^0}) : \deg_f(\tau) > 1\}$, which is a finite set since $\mathbf{V}^0$ is finite. For each $\tau \in \mathcal{P}^{>}(f|_{\mathbf{V}^0})$, we define

$$\eta_\tau \coloneqq \deg_f(\tau) e^{-s_0 l_{f,\phi}(\tau)},$$

and let

$$\eta \coloneqq \max_{\tau\in\mathcal{P}^{>}(f|_{\mathbf{V}^0})} \eta_\tau^{1/|\tau|}.$$

A key claim in the proof of Theorem D in [LZ24a, p. 82][2] establishes that $\eta_\tau < 1$ for all $\tau \in \mathcal{P}^{>}(f|_{\mathbf{V}^0})$, which implies that $\eta \in (0,1)$.

Now, consider a point $x \in \operatorname{Fix}(f^n)$ with $\deg_{f^n}(x) > 1$. Then x belongs to a primitive periodic orbit $\tau_x \in \mathcal{P}^{>}(f|_{\mathbf{V}^0})$ of period $k \coloneqq |\tau_x|$, and n must be a multiple of k, say $n = mk$ for some $m \in \mathbb{N}$. Thus $\deg_{f^n}(x) = (\deg_f(\tau_x))^m$ and $S_n\phi(x) = ml_{f,\phi}(\tau_x)$, which yields

$$\deg_{f^n}(x) e^{-s_0 S_n\phi(x)} = \bigl(\deg_f(\tau_x) e^{-s_0 l_{f,\phi}(\tau_x)}\bigr)^m = \eta^m_{\tau_x} \leqslant (\eta^k)^m = \eta^n.$$

This implies that

$$\begin{aligned}\bigl|\mathrm{I}_n(s)\bigr| &\leqslant \sum_{x\in\operatorname{Fix}(f^n)} \bigl(\deg_{f^n}(x) - 1\bigr) e^{-s_0 S_n\phi(x)} \\ &= \sum_{\substack{x\in\operatorname{Fix}(f^n)\cap\mathbf{V}^0\\ \deg_{f^n}(x)>1}} \bigl(\deg_{f^n}(x) - 1\bigr) e^{-s_0 S_n\phi(x)} \leqslant \operatorname{card}(\mathbf{V}^0)\eta^n.\end{aligned}$$

This establishes a desired exponential bound for $|\mathrm{I}_n(s)|$.

We next estimate $|\Pi_n(s)|$. Recall from Theorem 4.6 that $P(f|_{\mathbf{V}^0}, -s_0\phi|_{\mathbf{V}^0}) < P(f, -s_0\phi) = 0$ and

$$P(\sigma_{A_{\shortparallel}}, -s_0\phi\circ\pi_{\shortmid}\circ\pi_{\shortparallel}) = P(\sigma_{A_{\shortmid}}, -s_0\phi\circ\pi_{\shortmid}) = P(f|_{\mathcal{C}}, -s_0\phi|_{\mathcal{C}}) < P(f, -s_0\phi) = 0.$$

This allows us to choose $\delta > 0$ sufficiently small such that

$$P(f|_{\mathcal{C}}, -s_0\phi|_{\mathcal{C}}) + \delta < -\delta \quad\text{and}\quad P(f|_{\mathbf{V}^0}, -s_0\phi|_{\mathbf{V}^0}) + \delta < -\delta.$$

[2]The proof relies on the assumption that no 1-tile in $\mathbf{X}^1(f,\mathcal{C})$ joins opposite sides of $\mathcal{C}$.

Thus by Lemma 4.9 and (2.1), there exist constants $C_1, C_2, C_3 > 0$ such that for all $n \in \mathbb{N}$,

$$\begin{aligned}|\Pi_n(s)| &\leqslant \sum_{y \in \operatorname{Fix}(\sigma^n_{A_{\shortparallel}})} e^{-s_0 S_n \phi \circ \pi_{\shortmid} \circ \pi_{\shortparallel}(y)} + \sum_{y \in \operatorname{Fix}(\sigma^n_{A_{\shortmid}})} e^{-s_0 S_n \phi \circ \pi_{\shortmid}(y)} + \sum_{x \in \operatorname{Fix}((f|_{\mathbf{V}^0})^n)} e^{-s_0 S_n \phi(x)} \\ &\leqslant C_1 e^{n(P(\sigma_{A_{\shortparallel}}, -s_0 \phi \circ \pi_{\shortmid} \circ \pi_{\shortparallel}) + \delta)} + C_2 e^{n(P(\sigma_{A_{\shortmid}}, -s_0 \phi \circ \pi_{\shortmid}) + \delta)} + C_3 e^{n(P(f|_{\mathbf{V}^0}, -s_0 \phi|_{\mathbf{V}^0}) + \delta)} \\ &= (C_1 + C_2) e^{n(P(f|_{\mathcal{C}}, -s_0 \phi|_{\mathcal{C}}) + \delta)} + C_3 e^{n(P(f|_{\mathbf{V}^0}, -s_0 \phi|_{\mathbf{V}^0}) + \delta)} \\ &\leqslant (C_1 + C_2 + C_3) e^{-\delta n}.\end{aligned}$$

This establishes a desired exponential bound for $|\Pi_n(s)|$.

Combining the estimates for $|\mathrm{I}_n(s)|$ and $|\Pi_n(s)|$, we set $\kappa \coloneqq \max\{\eta, e^{-\delta}\} \in (0,1)$ and $C \coloneqq \operatorname{card}(\mathbf{V}^0) + C_1 + C_2 + C_3$. It follows that

$$\bigl|Z^{(n)}_{f,\phi}(s) - Z^{(n)}_{\sigma_{A_\vartriangle}, \phi \circ \pi_\vartriangle}(s)\bigr| \leqslant |\mathrm{I}_n(s)| + |\Pi_n(s)| \leqslant C\kappa^n,$$

which completes the proof. □

Proposition 4.11. *Let f, $\mathcal{C}$, d, Λ, β, ϕ, s_0, α satisfy the Assumptions in Section 3. We assume that ϕ satisfies the β-strong non-integrability condition, and that $f(\mathcal{C}) \subseteq \mathcal{C}$ and no 1-tile in $\mathbf{X}^1(f,\mathcal{C})$ joins opposite sides of $\mathcal{C}$. Define $\overline{\phi} \coloneqq \phi - \alpha$. Then for each $\varepsilon > 0$, there exist constants $T > 1$, $\rho \in (0,1)$, and $C > 0$ such that for each $t \in \mathbb{R} \smallsetminus (-T, T)$ and each integer $n \geqslant 2$, we have*

$$\bigl|Z^{(n)}_{f,\overline{\phi}}(-s_0 + \mathbf{i}t)\bigr| \leqslant C|t|^{2+\varepsilon} \rho^n e^{nP(f, -s_0\overline{\phi})}.$$

Proof. Let $s \coloneqq -s_0 + \mathbf{i}t$ with $t \in \mathbb{R}$. By Definition 2.14, we have the relation $Z^{(n)}_{f,\overline{\phi}}(s) = e^{-s\alpha n} Z^{(n)}_{f,\phi}(s)$. Note that $P(f, -s_0\overline{\phi}) = P(f, -s_0\phi) + s_0\alpha = s_0\alpha$ since $P(f, -s_0\phi) = 0$. Thus $\bigl|Z^{(n)}_{f,\overline{\phi}}(s)\bigr| = e^{nP(f,-s_0\overline{\phi})} \bigl|Z^{(n)}_{f,\phi}(s)\bigr|$ and it suffices to find an appropriate bound for $|Z^{(n)}_{f,\phi}(s)|$.

Let $\varepsilon > 0$ be arbitrary. Then by Proposition 4.5, there exist $T > 1$, $C_\vartriangle > 0$, and $\rho_\vartriangle \in (0,1)$ such that for all $n \geqslant 2$ and $t \in \mathbb{R} \smallsetminus (-T, T)$,

$$|Z^{(n)}_{\sigma_{A_\vartriangle}, \phi \circ \pi_\vartriangle}(s)| \leqslant C_\vartriangle |t|^{2+\varepsilon} \rho^n_\vartriangle.$$

By Lemma 4.10, there exist constants $\kappa \in (0,1)$ and $C_1 > 0$ such that $\bigl|Z^{(n)}_{f,\phi}(s) - Z^{(n)}_{\sigma_{A_\vartriangle}, \phi\circ\pi_\vartriangle}(s)\bigr| \leqslant C_1\kappa^n$ for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$. Let $\rho \coloneqq \max\{\rho_\vartriangle, \kappa\}$. Then for all $n \geqslant 2$ and $t \in \mathbb{R} \smallsetminus (-T, T)$, we have

$$\begin{aligned}\bigl|Z^{(n)}_{f,\phi}(s)\bigr| &\leqslant \bigl|Z^{(n)}_{\sigma_{A_\vartriangle}, \phi\circ\pi_\vartriangle}(s)\bigr| + \bigl|Z^{(n)}_{f,\phi}(s) - Z^{(n)}_{\sigma_{A_\vartriangle}, \phi\circ\pi_\vartriangle}(s)\bigr| \\ &\leqslant C_\vartriangle |t|^{2+\varepsilon}\rho^n_\vartriangle + C_1\kappa^n \leqslant (C_\vartriangle + C_1)|t|^{2+\varepsilon}\rho^n.\end{aligned}$$

Setting $C \coloneqq C_\vartriangle + C_1$ completes the proof. □

We recall the definition of the non-lattice property, which is needed to apply the complex Ruelle–Perron–Frobenius theorem in [PP90].

Definition 4.12. Consider a finite set of states S and a transition matrix $A \colon S \times S \to \{0,1\}$. Let $\bigl(\Sigma^+_A, \sigma_A\bigr)$ be the one-sided subshift of finite type defined by A. A real-valued function $\psi \colon \Sigma^+_A \to \mathbb{R}$ is called *non-lattice* if there exists no continuous function $u \colon \Sigma^+_A \to 2\pi\mathbb{Z}$ such that

$$\psi = C + u + v \circ \sigma_A - v$$

for some constant $C \in \mathbb{R}$ and continuous function $v \in C\bigl(\Sigma^+_A\bigr)$.

The following proposition is a part of [LZ24a, Theorem F].

Proposition 4.13 (Li & Zheng [LZ24a])**.** *Let $f \colon S^2 \to S^2$ be an expanding Thurston map and d a visual metric on S^2 for f. Let $\psi \in C^{0,\beta}\bigl((S^2,d), \mathbb{C}\bigr)$ be a complex-valued Hölder continuous function with an exponent $\beta \in (0,1]$. Then the following statements are equivalent:*

(i) *The function* ψ *is cohomologous to a constant in* $C(S^2,\mathbb{C})$, *i.e.,* $\psi = K + u\circ f - u$ *for some* $K\in\mathbb{C}$ *and* $u\in C(S^2,\mathbb{C})$.

(ii) *There exists* $n\in\mathbb{N}$ *and a Jordan curve* $\mathcal{C}\subseteq S^2$ *with* $f^n(\mathcal{C})\subseteq\mathcal{C}$ *and* $\operatorname{post} f\subseteq\mathcal{C}$ *such that the following statement holds for* $F\coloneqq f^n$, $\Psi\coloneqq S_n^f\psi$, *the one-sided subshift of finite type* $\bigl(\Sigma_{A_\Delta}^+,\sigma_{A_\Delta}\bigr)$ *associated to* F *and* $\mathcal{C}$ *defined in Proposition 2.6, and the factor map* $\pi_\Delta\colon\Sigma_{A_\Delta}^+\to S^2$ *defined in* (2.8):

The function $\Psi\circ\pi_\Delta$ *is cohomologous to a constant multiple of an integer-valued continuous function in* $C\bigl(\Sigma_{A_\Delta}^+,\mathbb{C}\bigr)$, *i.e.,* $\Psi\circ\pi_\Delta = KM + \varpi\circ\sigma_{A_\Delta} - \varpi$ *for some* $K\in\mathbb{C}$, $M\in C\bigl(\Sigma_{A_\Delta}^+,\mathbb{Z}\bigr)$, *and* $\varpi\in C\bigl(\Sigma_{A_\Delta}^+,\mathbb{C}\bigr)$.

This proposition is used in Lemma 4.15 (ii) to apply the complex Ruelle–Perron–Frobenius theorem [Pol84, Theorem 2].

Theorem 4.14 (Parry & Pollicott [PP90, Theorem 2(ii)])**.** *Consider a finite set of states* S *and a transition matrix* $A\colon S\times S\to\{0,1\}$. *Let* $\bigl(\Sigma_A^+,\sigma_A\bigr)$ *be the one-sided subshift of finite type defined by* A. *Fix* $\tau\in(0,1)$ *and equip the space* Σ_A^+ *with the metric* d_τ *defined in* (2.6).

Consider a complex-valued Hölder continuous function $\psi = u + iv \in C^{0,\beta}\bigl(\bigl(\Sigma_A^+,d_\tau\bigr),\mathbb{C}\bigr)$ *with* $u,v\in C^{0,\beta}\bigl(\Sigma_A^+,d_\tau\bigr)$. *If* $\bigl(\Sigma_A^+,\sigma_A\bigr)$ *is topologically mixing and* v *is non-lattice, then the spectrum of* $\mathcal{L}_\psi$ *is contained in a disc of radius strictly smaller than* $e^{P(\sigma,u)}$.

Applying the complex Ruelle–Perron–Frobenius theorem (Theorem 4.14) to the subshift $\bigl(\Sigma_{A_\Delta}^+,\sigma_{A_\Delta}\bigr)$ yields the following estimates for a bounded imaginary part.

Lemma 4.15. *Let* f, $\mathcal{C}$, d, Λ, β, ϕ, s_0 *satisfy the Assumptions in Section 3. We assume that* $f(\mathcal{C})\subseteq\mathcal{C}$. *Let* $\bigl(\Sigma_{A_\Delta}^+,\sigma_{A_\Delta}\bigr)$ *be the one-sided subshift of finite type associated to* f *and* $\mathcal{C}$ *defined in Proposition 2.6, and let* $\pi_\Delta\colon\Sigma_{A_\Delta}^+\to S^2$ *be defined in* (2.8). *Fix* $\tau\in(0,1)$ *and equip the space* $\Sigma_{A_\Delta}^+$ *with the metric* d_τ *defined in* (2.6). *Then the following statements hold:*

(i) *There exist constants* $t_0>0$, $\theta\in(0,1)$, *and* $C\geqslant 0$ *such that* $P(\sigma_{A_\Delta},(-s_0+\mathbf{i}t)\phi\circ\pi_\Delta)$ *is well-defined and*

$$\Bigl|Z^{(n)}_{\sigma_{A_\Delta},\,\phi\circ\pi_\Delta}(-s_0+\mathbf{i}t) - \exp\bigl(nP(\sigma_{A_\Delta},(-s_0+\mathbf{i}t)\phi\circ\pi_\Delta)\bigr)\Bigr| \leqslant C\theta^n$$

for all $t\in(-t_0,t_0)$ *and all* $n\in\mathbb{N}$.

(ii) *If* ϕ *is not cohomologous to a constant in* $C(S^2)$, *then for each* $\varepsilon>0$ *and each compact set* $K\subseteq\mathbb{R}$, *there exist* $\vartheta\in(0,1)$ *and* $C\geqslant 0$ *such that*

$$\Bigl|Z^{(n)}_{\sigma_{A_\Delta},\,\phi\circ\pi_\Delta}(-s_0+\mathbf{i}t)\Bigr| \leqslant C\vartheta^n$$

for all $t\in K\smallsetminus(-\varepsilon,\varepsilon)$ *and all* $n\in\mathbb{N}$.

Proof. (i) Note that $P(\sigma_{A_\Delta},(-s_0+\mathbf{i}t)\phi\circ\pi_\Delta)$ is well-defined when $|t|$ is sufficiently small. The statement follows from the argument in the proof of [PP90, Theorem 5.5 (ii)].

(ii) Assume that ϕ is not cohomologous to a constant in $C(S^2)$. Consider $t\in K\smallsetminus(-\varepsilon,\varepsilon)$ and denote $\psi_t\coloneqq(-s_0+\mathbf{i}t)\phi\circ\pi_\Delta$. Set $\delta\coloneqq\frac{1}{2}\log(\tau^{-\beta})>0$.

Since $P(\sigma_{A_\Delta},-s_0\phi\circ\pi_\Delta) = P(f,-s_0\phi) = 0$, by applying Corollary A.4 to the subshift $\bigl(\Sigma_{A_\Delta}^+,\sigma_{A_\Delta}\bigr)$ and the potential $\phi\circ\pi_\Delta$, we obtain that

$$\begin{aligned}\Bigl|Z^{(n)}_{\sigma_{A_\Delta},\,\phi\circ\pi_\Delta}(-s_0+\mathbf{i}t)\Bigr| &\leqslant C_\delta|t|\sum_{m=1}^{n}\bigl\|\mathcal{L}_{\psi_t}^{n-m}\bigr\|_{C^{0,\beta}}\bigl(\tau^\beta e^{P(\sigma_{A_\Delta},-s_0\phi\circ\pi_\Delta)+\delta}\bigr)^m\\ &= C_\delta|t|\sum_{m=1}^{n}\bigl\|\mathcal{L}_{\psi_t}^{n-m}\bigr\|_{C^{0,\beta}}\tau^{\beta m/2}\end{aligned}\tag{4.12}$$

for some constant $C_\delta>0$ depending only on f, $\mathcal{C}$, d, β, ϕ, b_0, τ, and δ.

We now bound the operator norm $\|\mathcal{L}_{\psi_t}^k\|_{C^{0,\beta}}$ for $k \in \mathbb{N}$ and $t \in K \smallsetminus (-\varepsilon, \varepsilon)$ by establishing a uniform estimate on the spectral radius of $\mathcal{L}_{\psi_t}$. Since ϕ is not cohomologous to a constant, Proposition 4.13 and Theorem 4.14 together imply that the spectral radius of $\mathcal{L}_{\psi_t}$, denoted by $\rho(\mathcal{L}_{\psi_t})$, is strictly less than $e^{P(\sigma_{A_\Delta}, \mathrm{Re}(\psi_t))} = 1$ for each $t \in K \smallsetminus (-\varepsilon, \varepsilon)$. Straightforward calculations show that $t \mapsto \mathcal{L}_{\psi_t}$ is continuous (as a map from $\mathbb{R}$ to the space of bounded linear operators on $C^{0,\beta}\big((\Sigma_A^+, d_\tau), \mathbb{C}\big)$). A classical result from operator theory asserts that the spectrum is upper semi-continuous on the space of bounded linear operators on a Banach space (cf. [Kat95, Remark 3.3, pp. 208–209]). It follows that the function $t \mapsto \rho(\mathcal{L}_{\psi_t})$ is upper semi-continuous, and hence attains its maximum on the compact set $K \smallsetminus (-\varepsilon, \varepsilon)$. Thus we have

$$\rho_0 := \max_{t \in K \smallsetminus (-\varepsilon, \varepsilon)} \rho(\mathcal{L}_{\psi_t}) < 1.$$

According to the spectral radius formula $\rho(\mathcal{L}_{\psi_t}) = \lim_{n\to\infty} \|\mathcal{L}_{\psi_t}^n\|_{C^{0,\beta}}^{1/n}$, we can find constants $\rho \in (\rho_0, 1)$ and $C_\rho \geqslant 1$ such that

$$\|\mathcal{L}_{\psi_t}^k\|_{C^{0,\beta}} \leqslant C_\rho \, \rho^k \quad \text{for all } k \in \mathbb{N} \text{ and } t \in K \smallsetminus (-\varepsilon, \varepsilon). \tag{4.13}$$

Substituting the estimate (4.13) into (4.12), we get

$$\Big| Z^{(n)}_{\sigma_{A_\Delta}, \, \phi \circ \pi_\Delta}(-s_0 + \mathbf{i}t) \Big| \leqslant C_\delta C_\rho |t| \sum_{m=1}^{n} \rho^{n-m} \tau^{\beta m/2} \leqslant C' n \big(\max\{\rho, \tau^{\beta/2}\}\big)^n,$$

where $C' := C_\delta C_\rho \sup_{t \in K} |t|$. Therefore, by setting $\vartheta := \frac{1}{2}\big(1 + \max\{\rho, \tau^{\beta/2}\}\big) \in (0,1)$ and $C := C' \sup_{n \in \mathbb{N}} \{n \vartheta^n\} < +\infty$, we establish the desired estimate. $\square$

Combining Lemmas 4.15 and 4.10, we obtain the following estimates for the partition function in the case of bounded imaginary part.

Lemma 4.16. *Let f, d, ϕ, s_0, α satisfy the Assumptions in Section 3. We assume that $f(\mathcal{C}) \subseteq \mathcal{C}$ and no 1-tile in $\mathbf{X}^1(f, \mathcal{C})$ joins opposite sides of $\mathcal{C}$. Define $\overline{\phi} := \phi - \alpha$. Then the following statements hold:*

(i) *There exist constants $t_0 > 0$, $\theta \in (0,1)$, and $C \geqslant 0$ such that $P(\sigma_{A_\Delta}, (-s_0 + \mathbf{i}t)\overline{\phi})$ is well-defined and*
$$\Big| Z^{(n)}_{f, \overline{\phi}}(-s_0 + \mathbf{i}t) - e^{nP(f, (-s_0 + \mathbf{i}t)\overline{\phi})} \Big| \leqslant C \theta^n e^{nP(f, -s_0 \overline{\phi})}$$
for all $t \in (-t_0, t_0)$ and all $n \in \mathbb{N}$.

(ii) *If ϕ is not cohomologous to a constant in $C(S^2)$, then for each $\varepsilon > 0$ and each compact set $K \subseteq \mathbb{R}$, there exist constants $\vartheta \in (0,1)$ and $C \geqslant 0$ such that*
$$\big| Z^{(n)}_{f, \overline{\phi}}(-s_0 + \mathbf{i}t) \big| \leqslant C \vartheta^n e^{nP(f, -s_0 \overline{\phi})}.$$
for all $t \in K \smallsetminus (-\varepsilon, \varepsilon)$ and all $n \in \mathbb{N}$.

The proof of Lemma 4.16 parallels that of Proposition 4.11, substituting the estimates from Lemma 4.15 for those from Proposition 4.5; the details are omitted.

5. Proof of the main theorem

In this section, we complete the proof of Theorem 1.1 using the results from the previous section. In Subsection 5.1, we introduce the notation and assumptions that are used throughout this section. In Subsection 5.2, we establish some auxiliary estimates (Lemma 5.1 and Proposition 5.2). In Subsection 5.3, we use these estimates to establish our main result.

This section is dedicated to the proof of Theorem 1.1. The argument follows a standard strategy for such counting problems and is organized into three main steps. First, in Subsection 5.2, we replace the sharp count of orbits with a smoothed version using a test function and relate this to a sum over all periodic points. Second, we employ Fourier analysis and the decay estimates from

Section 4 to establish the asymptotic behavior of this smoothed count. Finally, in Subsection 5.3, we complete the proof by using an approximation argument, where the sharp indicator function is bounded by smooth functions from above and below.

5.1. **Notation and assumptions.** Throughout this section, let f, $\mathcal{C}$, d, ϕ, s_0, α satisfy the Assumptions in Section 3. We assume that $f(\mathcal{C}) \subseteq \mathcal{C}$ and no 1-tile in $\mathbf{X}^1(f,\mathcal{C})$ joins opposite sides of $\mathcal{C}$. Suppose that ϕ is not cohomologous to a constant in $C(S^2)$. We define $\overline{\phi} \coloneqq \phi - \alpha$. Let $\sigma > 0$ be defined as in (4.3). Let $K \subseteq \mathbb{R}$ be a compact set and $\{I_n\}_{n\in\mathbb{N}}$ be a sequence of intervals contained in K.

For each $n \in \mathbb{N}$, we denote by p_n the midpoint of the interval I_n and by ℓ_n the length of I_n. Moreover, we assume that $\{\ell_n^{-1}\}_{n\in\mathbb{N}}$ has sub-exponential growth. Then we can write

$$\pi_{f,\phi}(n;\alpha,I_n) = \sum_{\tau\in\mathcal{P}_n(f)} \mathbb{1}_{I_n}\bigl(l_{f,\phi}(\tau) - n\alpha\bigr) = \sum_{\tau\in\mathcal{P}_n(f)} \mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}\bigl(\ell_n^{-1}\bigl(l_{f,\phi}(\tau) - n\alpha - p_n\bigr)\bigr), \tag{5.1}$$

where $l_{f,\phi}(\tau) = \sum_{y\in\tau}\phi(y)$.

5.2. **Auxiliary estimates.** In this subsection, we fix a non-negative function $\psi \in C^4(\mathbb{R},\mathbb{R}_{\geqslant 0})$ with compact support. For each $n \in \mathbb{N}$ we consider the auxiliary counting number

$$\pi_\psi(n) \coloneqq \sum_{\tau\in\mathcal{P}_n(f)} \psi\bigl(\ell_n^{-1}\bigl(l_{f,\phi}(\tau) - n\alpha - p_n\bigr)\bigr). \tag{5.2}$$

We study the asymptotic behavior of $\pi_\psi(n)$ to establish our main result by using an approximation argument in Subsection 5.3.

We start by transforming the summation over $\mathcal{P}_n(f)$, which represents the primitive periodic orbits of period n, into a summation over the set of fixed points of the iterated map f^n. Each primitive periodic orbit corresponds to n distinct points in this set. However, this set also contains points belonging to primitive periodic orbits of shorter lengths. The following lemma establishes a bound for the error introduced by these shorter primitive periodic orbits. For each $n \in \mathbb{N}$, define

$$\widetilde{\pi}_\psi(n) \coloneqq \frac{1}{n}\sum_{f^n(x)=x} \psi\bigl(\ell_n^{-1}\bigl(S_n^f\phi(x) - n\alpha - p_n\bigr)\bigr). \tag{5.3}$$

Lemma 5.1. *Consider a non-negative function $\psi \in C^4(\mathbb{R},\mathbb{R}_{\geqslant 0})$ with compact support. Following the assumptions in Subsection 5.1, we have that for each $\eta > 0$,*

$$\pi_\psi(n) = \widetilde{\pi}_\psi(n) + \mathcal{O}\Bigl(e^{(P(f,-s_0\overline{\phi})+\eta)n/2}\Bigr) \quad \text{as } n \to +\infty.$$

Here $\pi_\psi(n)$ and $\widetilde{\pi}_\psi(n)$ are defined by (5.2) *and* (5.3)*, respectively.*

The proof follows that of [SS22, Lemma 4.1]; we include it for completeness.

Proof. We say that a fixed point x of f^n is non-primitive if there exists a proper divisor q of n such that $f^q(x) = x$. Then we have

$$\begin{aligned}
\widetilde{\pi}_\psi(n) - \pi_\psi(n) &= \frac{1}{n}\sum_{\substack{f^n(x)=x\\ \text{non-primitive}}} \psi\bigl(\ell_n^{-1}\bigl(S_n^f\phi(x) - n\alpha - p_n\bigr)\bigr)\\
&= \frac{1}{n}\sum_{\substack{q\mid n\\ q\leqslant n/2}}\sum_{f^q(x)=x} \psi\bigl(\ell_n^{-1}\bigl(S_n^f\phi(x) - n\alpha - p_n\bigr)\bigr)\\
&= \frac{1}{n}\sum_{\substack{q\mid n\\ q\leqslant n/2}}\sum_{f^q(x)=x} \frac{\psi\bigl(\ell_n^{-1}\bigl(S_n^f\overline{\phi}(x) - p_n\bigr)\bigr)}{e^{-s_0S_q^f\overline{\phi}(x)}}e^{-s_0S_q^f\overline{\phi}(x)}.
\end{aligned}$$

Since ψ has compact support, it suffices to consider periodic points which satisfy $\ell_n^{-1}\big(S_n^f\overline{\phi}(x)-p_n\big)\in\operatorname{supp}\psi$, i.e., $S_n^f\overline{\phi}(x)\in p_n+\ell_n\operatorname{supp}\psi$. In particular, $S_q^f\overline{\phi}(x)$ is bounded for these periodic points (recall from Subsection 5.1 that the intervals I_n are contained in a compact set $K\subseteq\mathbb{R}$). Hence, for a non-primitive periodic point x satisfying $f^q(x)=x$ for q in the summation above, we get that $S_q^f\overline{\phi}(x)=\frac{q}{n}S_n^f\overline{\phi}(x)$ and thus $e^{-s_0S_q^f\overline{\phi}(x)}$ is uniformly bounded away from 0 for q in the summation above. Applying Lemma 4.16 (i), we deduce that for each $\eta>0$,

$$\begin{aligned}&\frac{1}{n}\sum_{\substack{q\mid n\\ q\leqslant n/2}}\sum_{f^q(x)=x}\frac{\psi\big(\ell_n^{-1}\big(S_n^f\overline{\phi}(x)-p_n\big)\big)}{e^{-s_0S_q^f\overline{\phi}(x)}}e^{-s_0S_q^f\overline{\phi}(x)}=\mathcal{O}\bigg(\frac{1}{n}\|\psi\|_\infty\sum_{q\leqslant n/2}\sum_{f^q(x)=x}e^{-s_0S_q^f\overline{\phi}(x)}\bigg)\\&=\mathcal{O}\bigg(\frac{1}{n}\|\psi\|_\infty\sum_{q\leqslant n/2}Z_{f,\overline{\phi}}^{(q)}(-s_0)\bigg)=\mathcal{O}\bigg(\frac{1}{n}\sum_{q\leqslant n/2}e^{(P(f,-s_0\overline{\phi})+\eta)q}\bigg)=\mathcal{O}\Big(e^{(P(f,-s_0\overline{\phi})+\eta)n/2}\Big)\end{aligned}$$

as $n\to+\infty$. This completes the proof. $\square$

For each $n\in\mathbb{N}$, we define

$$\psi_n(x):=\psi\big(\ell_n^{-1}(x-p_n)\big)e^{s_0(x-p_n)}. \tag{5.4}$$

Note that $\psi_n\in C^4(\mathbb{R},\mathbb{R}_{\geqslant0})$ has compact support. Recall that $\overline{\phi}=\phi-\alpha$. In this notation we have

$$\widetilde{\pi}_\psi(n)=\frac{1}{n}\sum_{f^n(x)=x}\psi_n\big(S_n^f\overline{\phi}(x)\big)e^{-s_0\left(S_n^f\overline{\phi}(x)-p_n\right)}. \tag{5.5}$$

We use Fourier transform to relate $\widetilde{\pi}_\psi(n)$ to partition functions so that we can apply the estimates established in Subsection 4.2.

Proposition 5.2. *Consider a non-negative function $\psi\in C^4(\mathbb{R},\mathbb{R}_{\geqslant0})$ with compact support. Under the assumptions in Subsection 5.1, we have that*

$$\widetilde{\pi}_\psi(n)\sim e^{s_0p_n}\frac{\int_{\mathbb{R}}\psi_n(x)\,\mathrm{d}x}{\sqrt{2\pi}\,\sigma}\frac{e^{P(f,-s_0\overline{\phi})n}}{n^{3/2}}\qquad\text{as } n\to+\infty.$$

Here $\widetilde{\pi}_\psi(n)$ is defined by (5.3) *and ψ_n is defined by* (5.4).

Proof. For each $n\in\mathbb{N}$ we define

$$A(n):=\bigg|\frac{\ell_n^{-1}e^{-s_0p_n}\sigma\sqrt{2\pi n^3}}{e^{P(f,-s_0\overline{\phi})n}}\widetilde{\pi}_\psi(n)\;-\;\ell_n^{-1}\int_{\mathbb{R}}\psi_n(x)\,\mathrm{d}x\bigg|.$$

The integral $\ell_n^{-1}\int_{\mathbb{R}}\psi_n(x)\,\mathrm{d}x=\int_{\mathbb{R}}\psi(y)e^{s_0\ell_ny}\,\mathrm{d}y$ is uniformly bounded away from 0 and $+\infty$ for $n\in\mathbb{N}$ as ψ has compact support. Hence it suffices to show that $A(n)\to0$ as $n\to+\infty$.

By applying the Fourier inversion theorem, for each $n\in\mathbb{N}$ we obtain

$$\psi_n(x)e^{-s_0(x-p_n)}=e^{s_0p_n}\int_{\mathbb{R}}\widehat{\psi}_n(t)e^{(-s_0+2\pi\mathbf{i}t)x}\,\mathrm{d}t. \tag{5.6}$$

Claim 1. For each $n\in\mathbb{N}$,

$$A(n)\leqslant\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\bigg|\frac{\ell_n^{-1}}{e^{P(f,-s_0\overline{\phi})n}}\widehat{\psi}_n\Big(\frac{t}{2\pi\sigma\sqrt{n}}\Big)Z_{f,\overline{\phi}}^{(n)}\Big(-s_0+\frac{\mathbf{i}t}{\sigma\sqrt{n}}\Big)\;-\;\ell_n^{-1}e^{-\frac{t^2}{2}}\int_{\mathbb{R}}\psi_n(x)\,\mathrm{d}x\bigg|\,\mathrm{d}t.$$

By (5.5) and (5.6), for each $n \in \mathbb{N}$ we have

$$\begin{aligned}
\frac{\ell_n^{-1} e^{-s_0 p_n} \sigma \sqrt{2\pi n^3}}{e^{P(f,-s_0\overline{\phi})n}} \widetilde{\pi}_\psi(n) &= \frac{\ell_n^{-1} e^{-s_0 p_n} \sigma \sqrt{2\pi n}}{e^{P(f,-s_0\overline{\phi})n}} \sum_{f^n(x)=x} \psi_n\bigl(S_n^f\overline{\phi}(x)\bigr) e^{-s_0\left(S_n^f\overline{\phi}(x)-p_n\right)} \\
&= \frac{\ell_n^{-1} \sigma \sqrt{2\pi n}}{e^{P(f,-s_0\overline{\phi})n}} \sum_{f^n(x)=x} \int_{\mathbb{R}} \widehat{\psi}_n(t) e^{(-s_0+2\pi \mathbf{i} t) S_n^f\overline{\phi}(x)} \,\mathrm{d}t \\
&= \frac{\ell_n^{-1}}{\sqrt{2\pi} e^{P(f,-s_0\overline{\phi})n}} \int_{\mathbb{R}} \widehat{\psi}_n\Bigl(\frac{t}{2\pi\sigma\sqrt{n}}\Bigr) \sum_{f^n(x)=x} e^{\left(-s_0+\frac{\mathbf{i}t}{\sigma\sqrt{n}}\right) S_n^f\overline{\phi}(x)} \,\mathrm{d}t \\
&= \frac{\ell_n^{-1}}{\sqrt{2\pi} e^{P(f,-s_0\overline{\phi})n}} \int_{\mathbb{R}} \widehat{\psi}_n\Bigl(\frac{t}{2\pi\sigma\sqrt{n}}\Bigr) Z_{f,\overline{\phi}}^{(n)}\Bigl(-s_0+\frac{\mathbf{i}t}{\sigma\sqrt{n}}\Bigr) \,\mathrm{d}t.
\end{aligned}$$

Here $Z_{f,\overline{\phi}}^{(n)}(\cdot)$ is defined in Definition 2.14. Using the identity $\sqrt{2\pi} = \int_{\mathbb{R}} e^{-t^2/2}\,\mathrm{d}t$ we deduce that

$$A(n) = \frac{1}{\sqrt{2\pi}} \biggl| \frac{\ell_n^{-1}}{e^{P(f,-s_0\overline{\phi})n}} \int_{\mathbb{R}} \widehat{\psi}_n\Bigl(\frac{t}{2\pi\sigma\sqrt{n}}\Bigr) Z_{f,\overline{\phi}}^{(n)}\Bigl(-s_0+\frac{\mathbf{i}t}{\sigma\sqrt{n}}\Bigr) \,\mathrm{d}t \;-\; \ell_n^{-1} \int_{\mathbb{R}} \psi_n(x)\,\mathrm{d}x \int_{\mathbb{R}} e^{-t^2/2}\,\mathrm{d}t \biggr|.$$

Then Claim 1 follows immediately from the integral inequality for absolute values.

We now consider the following three quantities:

$$\begin{aligned}
A_1(n) &:= \int_{|t|<\varepsilon\sigma\sqrt{n}} \biggl| \frac{\ell_n^{-1}}{e^{P(f,-s_0\overline{\phi})n}} \widehat{\psi}_n\Bigl(\frac{t}{2\pi\sigma\sqrt{n}}\Bigr) Z_{f,\overline{\phi}}^{(n)}\Bigl(-s_0+\frac{\mathbf{i}t}{\sigma\sqrt{n}}\Bigr) \;-\; \ell_n^{-1} e^{-\frac{t^2}{2}} \int_{\mathbb{R}} \psi_n(x)\,\mathrm{d}x \biggr| \,\mathrm{d}t, \\
A_2(n) &:= \int_{|t|\geqslant\varepsilon\sigma\sqrt{n}} \biggl| \frac{\ell_n^{-1}}{e^{P(f,-s_0\overline{\phi})n}} \widehat{\psi}_n\Bigl(\frac{t}{2\pi\sigma\sqrt{n}}\Bigr) Z_{f,\overline{\phi}}^{(n)}\Bigl(-s_0+\frac{\mathbf{i}t}{\sigma\sqrt{n}}\Bigr) \biggr| \,\mathrm{d}t, \\
A_3(n) &:= \int_{|t|\geqslant\varepsilon\sigma\sqrt{n}} \biggl| \ell_n^{-1} e^{-\frac{t^2}{2}} \int_{\mathbb{R}} \psi_n(x)\,\mathrm{d}x \biggr| \,\mathrm{d}t.
\end{aligned}$$

Here $\varepsilon \in (0,1)$ is chosen to be smaller than $\min\Bigl\{\delta, \frac{\sigma^2}{4C_\delta}, t_0\Bigr\}$, where the constants δ and C_δ are given by Lemma 4.2, and the constant t_0 is given by Lemma 4.16 (i). It follows from Claim 1 that

$$A(n) \leqslant \frac{1}{\sqrt{2\pi}} \bigl(A_1(n) + A_2(n) + A_3(n)\bigr) \qquad \text{for each } n \in \mathbb{N}.$$

Thus it suffices to show that $\lim_{n\to+\infty} A_i(n) = 0$ for each $i \in \{1,2,3\}$.

Claim 2. $\lim_{n\to+\infty} A_1(n) = 0$.

For each $n \in \mathbb{N}$, we define

$$\widetilde{A}_1(n) := \int_{|t|<\varepsilon\sigma\sqrt{n}} \biggl| \ell_n^{-1} \widehat{\psi}_n\Bigl(\frac{t}{2\pi\sigma\sqrt{n}}\Bigr) e^{n\left(P\left(f,(-s_0+\frac{\mathbf{i}t}{\sigma\sqrt{n}})\overline{\phi}\right)-P(f,-s_0\overline{\phi})\right)} \;-\; \ell_n^{-1} e^{-\frac{t^2}{2}} \int_{\mathbb{R}} \psi_n(x)\,\mathrm{d}x \biggr| \,\mathrm{d}t.$$

Then by Lemma 4.16 (i), there exists $\theta \in (0,1)$ such that as $n \to +\infty$,

$$A_1(n) = \widetilde{A}_1(n) + \mathcal{O}\bigl(\varepsilon\sigma\sqrt{n}\,\ell_n^{-1} \|\widehat{\psi}_n\|_\infty \theta^n\bigr),$$

where $\ell_n^{-1}\|\widehat{\psi}_n\|_\infty \leqslant \ell_n^{-1} \int_{\mathbb{R}} \psi_n(x)\,\mathrm{d}x = \int_{\mathbb{R}} \psi(y) e^{s_0 \ell_n y}\,\mathrm{d}y$ is uniformly bounded for $n \in \mathbb{N}$ since ψ has compact support. Hence, to establish Claim 2, it suffices to show that $\lim_{n\to+\infty} \widetilde{A}_1(n) = 0$.

By the triangle inequality we have that for each $n \in \mathbb{N}$,

$$\widetilde{A}_1(n) \leqslant \int_{|t|<\varepsilon\sigma\sqrt{n}} g_n(t)\,\mathrm{d}t + \int_{|t|<\varepsilon\sigma\sqrt{n}} h_n(t)\,\mathrm{d}t,$$

where for each $t \in \mathbb{R}$,

$$g_n(t) := \left| \ell_n^{-1} \widehat{\psi}_n\Big(\frac{t}{2\pi\sigma\sqrt{n}}\Big) e^{n\left(P\left(f,(-s_0+\frac{\mathbf{i}t}{\sigma\sqrt{n}})\overline{\phi}\right)-P(f,-s_0\overline{\phi})\right)} - \ell_n^{-1}\widehat{\psi}_n\Big(\frac{t}{2\pi\sigma\sqrt{n}}\Big)e^{-\frac{t^2}{2}} \right|,$$
$$h_n(t) := \left| e^{-\frac{t^2}{2}} \ell_n^{-1}\widehat{\psi}_n\Big(\frac{t}{2\pi\sigma\sqrt{n}}\Big) - e^{-\frac{t^2}{2}} \ell_n^{-1} \int_{\mathbb{R}} \psi_n(x)\,\mathrm{d}x \right|.$$

By (5.4) we have that

$$\ell_n^{-1}\widehat{\psi}_n\Big(\frac{t}{2\pi\sigma\sqrt{n}}\Big) = \ell_n^{-1}\int_{\mathbb{R}} \psi_n(x) e^{\frac{\mathbf{i}xt}{\sigma\sqrt{n}}}\,\mathrm{d}x = \int_{\mathbb{R}} \psi(y) e^{s_0\ell_n y} e^{\frac{\mathbf{i}t}{\sigma\sqrt{n}}(p_n+\ell_n y)}\,\mathrm{d}y,$$

which is uniformly bounded for $n \in \mathbb{N}$ and $t \in \mathbb{R}$ since ψ has compact support. Then it follows from Lebesgue's dominated convergence theorem that for each $t \in \mathbb{R}$,

$$0 \leqslant \lim_{n\to+\infty} h_n(t) \leqslant \lim_{n\to+\infty} e^{-\frac{t^2}{2}} \int_{\mathbb{R}} \psi(y) e^{s_0\ell_n y} \left| e^{\frac{\mathbf{i}t}{\sigma\sqrt{n}}(p_n+\ell_n y)} - 1 \right| \mathrm{d}y = 0.$$

Since $\int_{\mathbb{R}} e^{-t^2/2}\,\mathrm{d}t = \sqrt{2\pi} < +\infty$, by Lebesgue's dominated convergence theorem we conclude that $\lim_{n\to+\infty}\int_{\mathbb{R}} h_n(t)\,\mathrm{d}t = 0$. By (4.2) in Lemma 4.2, for each $t \in \mathbb{R}$,

$$\lim_{n\to+\infty} e^{n\left(P\left(f,(-s_0+\frac{\mathbf{i}t}{\sigma\sqrt{n}})\overline{\phi}\right)-P(f,-s_0\overline{\phi})\right)} = e^{-\frac{t^2}{2}}.$$

This implies that for each $t \in \mathbb{R}$, $\lim_{n\to+\infty} g_n(t) = 0$. Moreover, since $\varepsilon < \min\left\{\delta, \frac{\sigma^2}{4C_\delta}\right\}$, it follows from Lemma 4.2 that if $|t| < \varepsilon\sigma\sqrt{n}$, then

$$\left| e^{n\left(P\left(f,(-s_0+\frac{\mathbf{i}t}{\sigma\sqrt{n}})\overline{\phi}\right)-P(f,-s_0\overline{\phi})\right)} - e^{-\frac{t^2}{2}} \right| \leqslant e^{-\frac{t^2}{2}\left(1-\frac{2C_\delta t}{\sigma^3\sqrt{n}}\right)} + e^{-\frac{t^2}{2}} < e^{-\frac{t^2}{4}} + e^{-\frac{t^2}{2}}.$$

Hence, by Lebesgue's dominated convergence theorem, $\lim_{n\to+\infty}\int_{|t|<\varepsilon\sigma\sqrt{n}} g_n(t)\,\mathrm{d}t = 0$. This implies $\lim_{n\to+\infty}\widetilde{A}_1(n) = 0$ and establishes Claim 2.

Claim 3. $\lim_{n\to+\infty} A_2(n) = 0$.

Let $T > 1$ be the constant given by Proposition 4.11. By Lemma 4.16 (ii), there exists $\vartheta \in (0,1)$ such that

$$\int_{\varepsilon\leqslant\frac{|t|}{\sigma\sqrt{n}}\leqslant T} \left| \frac{\ell_n^{-1}}{e^{P(f,-s_0\overline{\phi})n}} \widehat{\psi}_n\Big(\frac{t}{2\pi\sigma\sqrt{n}}\Big) Z_{f,\overline{\phi}}^{(n)}\Big(-s_0 + \frac{\mathbf{i}t}{\sigma\sqrt{n}}\Big) \right| \mathrm{d}t = \mathcal{O}\big(\sigma\sqrt{n}\,\ell_n^{-1}\|\widehat{\psi}_n\|_\infty\vartheta^n\big) \tag{5.7}$$

as $n \to +\infty$, where $\ell_n^{-1}\|\widehat{\psi}_n\|_\infty \leqslant \ell_n^{-1}\int_{\mathbb{R}}\psi_n(x)\,\mathrm{d}x = \int_{\mathbb{R}}\psi(y)e^{s_0\ell_n y}\,\mathrm{d}y$ is uniformly bounded for $n \in \mathbb{N}$ since ψ has compact support. Since $\psi_n \in C^4(\mathbb{R},\mathbb{R})$, we have that for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$,

$$(2\pi\mathbf{i}t)^4\widehat{\psi}_n(t) = \widehat{\psi_n^{(4)}}(t).$$

Moreover, since ψ has compact support, we have that

$$\ell_n^3\left\|\widehat{\psi_n^{(4)}}\right\|_\infty \leqslant \ell_n^3\int_{\mathbb{R}}\big|\psi_n^{(4)}(x)\big|\,\mathrm{d}x = \int_{\mathbb{R}}\left|\sum_{k=0}^4\binom{4}{k}\ell_n^k\,\psi^{(4-k)}(y)(s_0)^k e^{s_0\ell_n y}\right|\mathrm{d}y \leqslant C'$$

for some constant $C' \geqslant 0$, which is independent of n. Then, by Proposition 4.11, there exist $C > 0$ and $\rho \in (0,1)$ such that for each integer $n \geqslant 2$,

$$\begin{aligned}
&\int_{|t|>T\sigma\sqrt{n}} \left| \frac{\ell_n^{-1}}{e^{P(f,-s_0\overline{\phi})n}} \widehat{\psi}_n\Big(\frac{t}{2\pi\sigma\sqrt{n}}\Big) Z_{f,\overline{\phi}}^{(n)}\Big(-s_0 + \frac{\mathbf{i}t}{\sigma\sqrt{n}}\Big) \right| \mathrm{d}t \\
&\quad\leqslant \int_{|t|>T\sigma\sqrt{n}} \left| \ell_n^{-1} \widehat{\psi}_n\Big(\frac{t}{2\pi\sigma\sqrt{n}}\Big) C \Big|\frac{t}{\sigma\sqrt{n}}\Big|^{2+\varepsilon} \rho^n \right| \mathrm{d}t \\
&\quad= \int_{|t|>T} C\sigma\sqrt{n}\, \ell_n^{-1} \rho^n |t|^{2+\varepsilon} \left| \widehat{\psi}_n\Big(\frac{t}{2\pi}\Big) \right| \mathrm{d}t \\
&\quad= C\sigma\sqrt{n}\, \ell_n^{-4} \rho^n \int_{|t|>T} |t|^{-2+\varepsilon}\, \ell_n^3 \left| \widehat{\psi_n^{(4)}}\Big(\frac{t}{2\pi}\Big) \right| \mathrm{d}t \\
&\quad\leqslant CC'\sigma\sqrt{n}\, \ell_n^{-4} \rho^n \int_{|t|>T} |t|^{-2+\varepsilon}\, \mathrm{d}t.
\end{aligned}$$

Combining this with (5.7) and recalling that the sequences $\{\ell_n^{-1}\}_{n\in\mathbb{N}}$ are of sub-exponential growth, we establish Claim 3.

Finally, it is clear that $\lim_{n\to+\infty} A_3(n) = 0$. We conclude the proof by combining the estimates in Claims 2 and 3. $\square$

The following corollary is an immediate consequence of Proposition 5.2 and Lemma 5.1. Recall that $P(f, -s_0\overline{\phi}) = s_0\alpha > 0$.

Corollary 5.3. *Consider a non-negative function $\psi \in C^4(\mathbb{R}, \mathbb{R}_{\geqslant 0})$ with compact support. Under the assumptions in Subsection 5.1, we have that*

$$\pi_\psi(n) \sim e^{s_0 p_n} \frac{\int_{\mathbb{R}} \psi_n(x)\,\mathrm{d}x}{\sigma\sqrt{2\pi}} \frac{e^{s_0\alpha n}}{n^{3/2}} \quad \textit{as } n \to +\infty.$$

Here $\pi_\psi(n)$ is defined by (5.2) *and ψ_n is defined by* (5.4).

5.3. **Approximation argument.** In this subsection we prove Theorem 1.1 by using the auxiliary estimates from Subsection 5.2 through an approximation argument.

Proof of Theorem 1.1. By Lemma 2.4, it suffices to prove the theorem for the case where $f(\mathcal{C}) \subseteq \mathcal{C}$ and no 1-tile in $\mathbf{X}^1(f, \mathcal{C})$ joins opposite sides of $\mathcal{C}$. Hence all the assumptions in Subsection 5.1 are satisfied.

For each $n \in \mathbb{N}$, we define

$$B(n) := \ell_n^{-1} e^{-s_0 p_n} \frac{\sigma\sqrt{2\pi n^3}}{e^{s_0\alpha n}} \pi_{f,\phi}(n; \alpha, I_n) \,-\, \ell_n^{-1} \int_{I_n} e^{s_0(z-p_n)}\,\mathrm{d}z.$$

The integral $\ell_n^{-1}\int_{I_n} e^{s_0(z-p_n)}\,\mathrm{d}z$ is uniformly bounded away from 0 and $+\infty$ for $n \in \mathbb{N}$ as K is compact. Thus, in order to prove this theorem, it suffices to show that $\lim_{n\to+\infty} B(n) = 0$.

Fix an arbitrary $\varepsilon \in (0,1)$.

We first construct a non-negative function $\psi \in C^4(\mathbb{R}, \mathbb{R}_{\geqslant 0})$ with compact support satisfying the following properties:

$$\mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]} \leqslant \psi \leqslant 1+\varepsilon, \quad \operatorname{supp}\psi \subseteq \Big[-\frac{1+\varepsilon}{2}, \frac{1+\varepsilon}{2}\Big], \quad \text{and} \quad \int_{\mathbb{R}} \psi(x)\,\mathrm{d}x \leqslant 1+\varepsilon.$$

Let $\eta \in C^\infty(\mathbb{R})$ be a non-negative mollifier defined by

$$\eta(x) := \begin{cases} C\exp\big(-\frac{1}{1-x^2}\big) & \text{if } |x| < 1; \\ 0 & \text{otherwise,} \end{cases}$$

where $C > 0$ is a normalization constant such that $\int_{\mathbb{R}} \eta(x)\,\mathrm{d}x = 1$. For each $\delta > 0$, set $\eta_\delta(x) := \delta^{-1}\eta(x/\delta)$. Consider $G := (1+\varepsilon/4)\mathbb{1}_{[-(1+\varepsilon/4)/2,\,(1+\varepsilon/4)/2]}$. Then there exists a sufficiently small $\delta > 0$ such that the function $\psi := G * \eta_\delta$ satisfies the desired properties. Here $*$ denotes the convolution of functions.

By (5.1), (5.2), and Corollary 5.3, we have that

$$\begin{aligned}
&\limsup_{n\to+\infty} B(n)\\
&= \limsup_{n\to+\infty} \ell_n^{-1} e^{-s_0 p_n} \frac{\sigma\sqrt{2\pi n^3}}{e^{s_0\alpha n}} \sum_{\tau\in\mathcal{P}_n(f)} \mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}\big(\ell_n^{-1}\big(l_{f,\phi}(\tau) - n\alpha - p_n\big)\big) \;-\; \ell_n^{-1}\int_{I_n} e^{s_0(z-p_n)}\,\mathrm{d}z\\
&\leqslant \limsup_{n\to+\infty} \ell_n^{-1} e^{-s_0 p_n} \frac{\sigma\sqrt{2\pi n^3}}{e^{s_0\alpha n}} \sum_{\tau\in\mathcal{P}_n(f)} \psi\big(\ell_n^{-1}\big(l_{f,\phi}(\tau) - n\alpha - p_n\big)\big) \;-\; \ell_n^{-1}\int_{I_n} e^{s_0(z-p_n)}\,\mathrm{d}z.\\
&= \limsup_{n\to+\infty} \ell_n^{-1}\int_{\mathbb{R}} \psi_n(x)\,\mathrm{d}x \;-\; \ell_n^{-1}\int_{I_n} e^{s_0(z-p_n)}\,\mathrm{d}z,
\end{aligned}$$

where ψ_n is defined by (5.4) and $\ell_n^{-1}\int_{\mathbb{R}}\psi_n(x)\,\mathrm{d}x = \int_{\mathbb{R}}\psi(y)e^{s_0\ell_n y}\,\mathrm{d}y$ is uniformly bounded for $n\in\mathbb{N}$ since ψ has compact support. For each $n\in\mathbb{N}$,

$$\begin{aligned}
\ell_n^{-1}\int_{\mathbb{R}}\psi_n(x)\,\mathrm{d}x &= \int_{\mathbb{R}}\psi(y)e^{s_0\ell_n y}\,\mathrm{d}y = \int_{-\frac{1+\varepsilon}{2}}^{\frac{1+\varepsilon}{2}}\psi(y)e^{s_0\ell_n y}\,\mathrm{d}y\\
&\leqslant \int_{-\frac{1}{2}}^{\frac{1}{2}}\psi(y)e^{s_0\ell_n y}\,\mathrm{d}y + \varepsilon(1+\varepsilon)e^{s_0|K|}\\
&\leqslant (1+\varepsilon)\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{s_0\ell_n y}\,\mathrm{d}y + \varepsilon(1+\varepsilon)e^{s_0|K|}\\
&= (1+\varepsilon)\ell_n^{-1}\int_{I_n} e^{s_0(z-p_n)}\,\mathrm{d}z + \varepsilon(1+\varepsilon)e^{s_0|K|}\\
&\leqslant \ell_n^{-1}\int_{I_n} e^{s_0(z-p_n)}\,\mathrm{d}z + \varepsilon(2+\varepsilon)e^{s_0|K|},
\end{aligned}$$

where $|K|$ denotes the diameter of the compact set K. Thus, we obtain that

$$\limsup_{n\to+\infty} B(n) \leqslant \varepsilon(2+\varepsilon)e^{s_0|K|} \leqslant C_1\varepsilon,$$

where the constant $C_1 := 3e^{s_0|K|}$ depends only on s_0 and K.

Similarly, one can show that

$$\liminf_{n\to+\infty} B(n) \geqslant -C_2\varepsilon$$

for some constant $C_2 > 0$ that depends only on $-s_0$ and K.

Since the choice of $\varepsilon\in(0,1)$ was arbitrary, we have that $\lim_{n\to+\infty} B(n) = 0$. This completes the proof. □

Appendix A. Ruelle lemma

This appendix presents a proof of the Ruelle lemma for one-sided subshifts of finite type (see Lemma A.2). Estimates of this type first appeared implicitly in [Rue90], and variations were established in [PS98, PS01, Nau05]. The lemma provides an estimate of the difference between the partition function and a sum involving iterates of the Ruelle operator acting on characteristic functions of cylinders. This estimate is crucial for analyzing the analytic properties of dynamical zeta functions and establishing statistical limit theorems.

Throughout this appendix, we fix the following notation. Consider a finite set of states S and a transition matrix $A\colon S\times S \to \{0,1\}$. Denote by $\left(\Sigma_A^+,\sigma_A\right)$ the one-sided subshift of finite type defined by A. Fix $\tau\in(0,1)$ and equip the space Σ_A^+ with the metric d_τ defined in (2.6). Let $\beta\in(0,1]$ and $\phi\in C^{0,\beta}\left(\Sigma_A^+,d_\tau\right)$ be a real-valued Hölder continuous function. We refer the reader to Subsection 2.4 for background on symbolic dynamics and the Ruelle operator.

We introduce definitions needed to state and prove the Ruelle lemma.

Definition A.1 (Admissible word, cylinder set, and concatenation)**.** For $n\in\mathbb{N}$ and a finite word $\omega=\omega_0\cdots\omega_{n-1}$ with $\omega_i\in S$ and $A(\omega_i,\omega_{i+1})=1$ for every $0\leqslant i<n-1$, we say ω is an *admissible word of length* n, denoted $|\omega|=n$. The *cylinder set* associated to ω is

$$C_\omega=[\omega]\coloneqq\left\{x=\{x_i\}_{i\in\mathbb{N}_0}\in\Sigma_A^+ : x_i=\omega_i \text{ for } 0\leqslant i<n\right\}.$$

We denote by χ_ω the characteristic function of the cylinder set C_ω. For a point $x\in\Sigma_A^+$ satisfying $A(\omega_{n-1},x_0)=1$, we write $\omega x\in\Sigma_A^+$ for the concatenation obtained by prepending ω to x; explicitly, $(\omega x)_i=\omega_i$ for $0\leqslant i<n$ and $(\omega x)_i=x_{i-n}$ for $i\geqslant n$.

We now state the Ruelle lemma.

Lemma A.2 (Ruelle lemma)**.** *For each $a_0>0$, each $b_0>0$, and each $\varepsilon>0$, there exists a constant $C_\varepsilon>0$ such that for all $k\in\mathbb{N}$, $n\in\mathbb{N}$ with $n\geqslant 2$, and $s\in\mathbb{C}$ with $|\mathrm{Re}(s)|\leqslant a_0$ and $|\mathrm{Im}(s)|\geqslant b_0$, we have*

$$\sum_{|\omega|=k}\left\|\mathcal{L}_{s\phi}^k\chi_\omega\right\|_{C^{0,\beta}}\leqslant C_\varepsilon|\mathrm{Im}(s)|\tau^\beta e^{k(P(\sigma,\mathrm{Re}(s)\phi)+\varepsilon)} \tag{A.1}$$

and

$$\left|\sum_{\sigma^n x=x}e^{sS_n\phi(x)}-\sum_{j\in S}\mathcal{L}_{s\phi}^n\chi_j(x_j)\right|\leqslant C_\varepsilon|\mathrm{Im}(s)|\sum_{m=2}^n\left\|\mathcal{L}_{s\phi}^{n-m}\right\|_{C^{0,\beta}}\left(\tau^\beta e^{P(\sigma,\mathrm{Re}(s)\phi)+\varepsilon}\right)^m \tag{A.2}$$

for any choice of a point $x_j\in C_j$ for each cylinder set C_j. Here the sum in (A.1) *is taken over all admissible words ω of length k, and $\left\|\mathcal{L}_{s\phi}^{n-m}\right\|_{C^{0,\beta}}$ denotes the operator norm of $\mathcal{L}_{s\phi}^{n-m}$ on $C^{0,\beta}\left(\left(\Sigma_A^+,d_\tau\right),\mathbb{C}\right)$.*

Remark. The constant C_ε in Lemma A.2 only depends on ε, a_0, b_0, τ, β, ϕ, and the ambient parameters of the subshift; it is independent of n and the specific value of s (as long as $|\mathrm{Re}(s)|\leqslant a_0$ and $|\mathrm{Im}(s)|\geqslant b_0$).

The following lemma expresses the partition function as a sum over the Ruelle operator acting on the characteristic functions of the cylinder sets.

Lemma A.3. *For each $\psi\in C(X,\mathbb{C})$ and each $n\in\mathbb{N}$ we have*

$$\sum_{x\in\mathrm{Fix}(\sigma^n)}e^{S_n\psi(x)}=\sum_{|\omega|=n}\left(\mathcal{L}_\psi^n\chi_\omega\right)(x_\omega),$$

where for each admissible word ω the point $x_\omega\in C_\omega$ is defined as follows: if the cylinder set C_ω contains a fixed point of σ^n, we take x_ω to be that point; otherwise we choose x_ω arbitrarily in C_ω.

Proof. Fix an arbitrary $n\in\mathbb{N}$. Since $\Sigma_A^+=\bigcup_{|\omega|=n}C_\omega$, we have

$$\sum_{x\in\mathrm{Fix}(\sigma^n)}e^{S_n\psi(x)}=\sum_{|\omega|=n}\ \sum_{x\in C_\omega\cap\mathrm{Fix}(\sigma^n)}e^{S_n\psi(x)}.$$

By definition of the Ruelle operator,

$$\left(\mathcal{L}_\psi^n\chi_\omega\right)(x_\omega)=\sum_{y\in C_\omega\cap\sigma^{-n}(x_\omega)}e^{S_n\psi(y)}.$$

Thus it suffices to show that $C_\omega \cap \mathrm{Fix}(\sigma^n) = C_\omega \cap \sigma^{-n}(x_\omega)$ for each admissible word ω with $|\omega| = n$.

Let ω be an arbitrary admissible word of length n. Note that the cylinder set C_ω can contain at most one point from each of the sets $\mathrm{Fix}(\sigma^n)$ and $\sigma^{-n}(x_\omega)$. We consider two cases.

If $A(\omega_{n-1}, \omega_0) = 1$, the word ω can be followed by itself, which forms the point $x^*_\omega := \omega\omega\cdots$. This point lies in C_ω and is a fixed point of σ^n. This implies that $x_\omega = x^*_\omega$ and

$$C_\omega \cap \mathrm{Fix}(\sigma^n) = C_\omega \cap \sigma^{-n}(x_\omega) = \{x_\omega\}.$$

On the other hand, if $A(\omega_{n-1}, \omega_0) = 0$, then the word ω cannot be followed by the symbol ω_0. This implies that

$$C_\omega \cap \mathrm{Fix}(\sigma^n) = C_\omega \cap \sigma^{-n}(x_\omega) = \emptyset.$$

In either case, the equality $C_\omega \cap \mathrm{Fix}(\sigma^n) = C_\omega \cap \sigma^{-n}(x_\omega)$ holds. This completes the proof. $\square$

We now prove Lemma A.2. The proof proceeds in three steps: establishing a telescoping sum identity, estimating the Hölder norms, and bounding the sum over admissible words.

Proof of Lemma A.2. Let $a_0 > 0$, $b_0 > 0$, and $\varepsilon > 0$ be arbitrary. Fix an arbitrary integer $n \geqslant 2$. Consider $s = a + \mathbf{i}b \in \mathbb{C}$ with $|a| \leqslant a_0$ and $|b| \geqslant b_0$, where $a := \mathrm{Re}(s)$ and $b := \mathrm{Im}(s)$.

For each admissible word ω of length n, we choose $x_\omega \in C_\omega$ in the following way: if the cylinder set C_ω contains a fixed point of σ^n, we take x_ω to be that point; otherwise we choose x_ω arbitrarily in C_ω. Next, for each $m \in \{1, \ldots, n-1\}$ and each admissible word ω of length m, we fix an arbitrary point $x_\omega \in C_\omega$.

Denote

$$Z_n(s) := \sum_{x \in \mathrm{Fix}(\sigma^n)} e^{sS_n\phi(x)} \quad \text{and} \quad T_n(s) := \sum_{j \in S} (\mathcal{L}^n_{s\phi}\chi_j)(x_j).$$

Claim.

$$Z_n(s) - T_n(s) = \sum_{m=2}^{n} \sum_{|\omega|=m} \big((\mathcal{L}^n_{s\phi}\chi_\omega)(x_\omega) - (\mathcal{L}^n_{s\phi}\chi_\omega)(x_{\widehat{\omega}}) \big),$$

where $\widehat{\omega}$ denotes the word obtained by removing the last symbol from ω.

Proof of the claim. By Lemma A.3, $Z_n(s) = \sum_{|\omega|=n} (\mathcal{L}^n_{s\phi}\chi_\omega)(x_\omega)$. For $m \in \{1, \ldots, n\}$, define

$$A_m := \sum_{|\omega|=m} (\mathcal{L}^n_{s\phi}\chi_\omega)(x_\omega).$$

Then $Z_n(s) - T_n(s) = A_n - A_1 = \sum_{m=2}^n (A_m - A_{m-1})$. Note that for each $m \in \{2, \ldots, n\}$ and each admissible word η of length $m-1$, the characteristic function satisfies

$$\chi_\eta = \sum_{j \in S \,:\, A(\eta_{m-2}, j)=1} \chi_{\eta j},$$

where ηj denotes the word obtained by appending j to η. Thus for each $m \in \{2, \ldots, n\}$, we have

$$\begin{aligned} A_{m-1} &= \sum_{|\eta|=m-1} (\mathcal{L}^n_{s\phi}\chi_\eta)(x_\eta) \\ &= \sum_{|\eta|=m-1} \sum_{j \in S \,:\, A(\eta_{m-2}, j)=1} (\mathcal{L}^n_{s\phi}\chi_{\eta j})(x_\eta) \\ &= \sum_{|\omega|=m} (\mathcal{L}^n_{s\phi}\chi_\omega)(x_{\widehat{\omega}}), \end{aligned}$$

where we set $\omega = \eta j$, so that $\widehat{\omega} = \eta$. It follows that

$$A_m - A_{m-1} = \sum_{|\omega|=m} \big((\mathcal{L}^n_{s\phi}\chi_\omega)(x_\omega) - (\mathcal{L}^n_{s\phi}\chi_\omega)(x_{\widehat{\omega}}) \big).$$

Summing this equality over $m \in \{2, \ldots, n\}$ yields the desired identity, establishing the claim.

By the claim, we have

$$
\begin{aligned}
|Z_n(s) - T_n(s)| &\leqslant \sum_{m=2}^{n} \sum_{|\omega|=m} |(\mathcal{L}_{s\phi}^n \chi_\omega)(x_\omega) - (\mathcal{L}_{s\phi}^n \chi_\omega)(x_{\widehat{\omega}})| \\
&\leqslant \sum_{m=2}^{n} \sum_{|\omega|=m} |\mathcal{L}_{s\phi}^n \chi_\omega|_\beta \, d_\tau(x_\omega, x_{\widehat{\omega}})^\beta \\
&\leqslant \sum_{m=2}^{n} \sum_{|\omega|=m} \|\mathcal{L}_{s\phi}^n \chi_\omega\|_{C^{0,\beta}} \, \tau^{\beta(m-1)}.
\end{aligned}
$$

Since $\mathcal{L}_{s\phi}^{n-m}$ is bounded on $C^{0,\beta}\big((\Sigma_A^+, d_\tau), \mathbb{C}\big)$ (cf. [PP90, Proposition 2.1]), we have

$$
\|\mathcal{L}_{s\phi}^n \chi_\omega\|_{C^{0,\beta}} = \|\mathcal{L}_{s\phi}^{n-m}\big(\mathcal{L}_{s\phi}^m \chi_\omega\big)\|_{C^{0,\beta}} \leqslant \big\|\mathcal{L}_{s\phi}^{n-m}\big\|_{C^{0,\beta}} \|\mathcal{L}_{s\phi}^m \chi_\omega\|_{C^{0,\beta}},
$$

where $\big\|\mathcal{L}_{s\phi}^{n-m}\big\|_{C^{0,\beta}}$ denotes the operator norm of $\mathcal{L}_{s\phi}^{n-m}$ on $C^{0,\beta}\big((\Sigma_A^+, d_\tau), \mathbb{C}\big)$. Hence,

$$
|Z_n(s) - T_n(s)| \leqslant \tau^{-\beta} \sum_{m=2}^{n} \tau^{\beta m} \big\|\mathcal{L}_{s\phi}^{n-m}\big\|_{C^{0,\beta}} \sum_{|\omega|=m} \|\mathcal{L}_{s\phi}^m \chi_\omega\|_{C^{0,\beta}}. \tag{A.3}
$$

We now bound the sum $\sum_{|\omega|=m} \|\mathcal{L}_{s\phi}^m \chi_\omega\|_{C^{0,\beta}}$ for all $m \in \mathbb{N}$. For each admissible word ω of length m, we have

$$
(\mathcal{L}_{s\phi}^m \chi_\omega)(x) = \sum_{y \in C_\omega \,:\, \sigma^m(y) = x} e^{sS_m\phi(y)} \qquad \text{for } x \in \Sigma_A^+.
$$

This implies that $\|\mathcal{L}_{s\phi}^m \chi_\omega\|_\infty \leqslant \sup_{y \in C_\omega} e^{aS_m\phi(y)}$. For each $x \in \Sigma_A^+$, there is at most one $y \in C_\omega$ satisfying $\sigma^m(y) = x$. When such a y exists, we denote it by y_x^ω. Thus,

$$
(\mathcal{L}_{s\phi}^m \chi_\omega)(x) = \begin{cases} e^{sS_m\phi(y_x^\omega)} & \text{if } y_x^\omega \text{ exists;} \\ 0 & \text{otherwise.} \end{cases}
$$

To estimate the Hölder seminorm $|\mathcal{L}_{s\phi}^m \chi_\omega|_\beta$, we consider distinct points $x = \{x_i\}_{i \in \mathbb{N}_0}$ and $z = \{z_i\}_{i \in \mathbb{N}_0}$ in Σ_A^+. According to the existence of y_x^ω and y_z^ω, we consider the following three cases.

Case 1: Neither y_x^ω nor y_z^ω exists. Then $(\mathcal{L}_{s\phi}^m \chi_\omega)(x) = (\mathcal{L}_{s\phi}^m \chi_\omega)(z) = 0$, and $|(\mathcal{L}_{s\phi}^m \chi_\omega)(x) - (\mathcal{L}_{s\phi}^m \chi_\omega)(z)| = 0$ holds trivially.

Case 2: Exactly one of y_x^ω and y_z^ω exists. In this case, we must have $d_\tau(x, z) = 1$ (i.e., $x_0 \neq z_0$); otherwise the condition $x_0 = z_0$ would imply that either both y_x^ω and y_z^ω exist, or neither exists, contradicting the assumption that exactly one of them exists. Then one of $(\mathcal{L}_{s\phi}^m \chi_\omega)(x)$ and $(\mathcal{L}_{s\phi}^m \chi_\omega)(z)$ vanishes, and we have

$$
\big|(\mathcal{L}_{s\phi}^m \chi_\omega)(x) - (\mathcal{L}_{s\phi}^m \chi_\omega)(z)\big| \leqslant \|\mathcal{L}_{s\phi}^m \chi_\omega\|_\infty \leqslant d_\tau(x, z)^\beta \sup_{y \in C_\omega} e^{aS_m\phi(y)}.
$$

Case 3: Both y_x^ω and y_z^ω exist. Since $y_x^\omega = \omega x$ and $y_z^\omega = \omega z$ by definition, we have $d_\tau(y_x^\omega, y_z^\omega) = d_\tau(x,z)\tau^m$. Using the Hölder continuity of ϕ, we deduce that

$$\begin{aligned}
|S_m\phi(y_x^\omega) - S_m\phi(y_z^\omega)| &\leqslant \sum_{i=0}^{m-1} |\phi(\sigma^i(y_x^\omega)) - \phi(\sigma^i(y_z^\omega))| \\
&\leqslant \sum_{i=0}^{m-1} \|\phi\|_{C^{0,\beta}} d_\tau\big(\sigma^i(y_x^\omega), \sigma^i(y_z^\omega)\big)^\beta \\
&= \|\phi\|_{C^{0,\beta}} d_\tau(y_x^\omega, y_z^\omega)^\beta \sum_{i=0}^{m-1} \tau^{-i\beta} \\
&= \|\phi\|_{C^{0,\beta}} d_\tau(x,z)^\beta \tau^{\beta m} \frac{\tau^{-\beta m} - 1}{\tau^{-\beta} - 1} \\
&\leqslant \frac{\|\phi\|_{C^{0,\beta}}}{\tau^{-\beta} - 1} d_\tau(x,z)^\beta.
\end{aligned}$$

Applying the inequality $|e^z - e^w| \leqslant |z - w|\, e^{\max\{\mathrm{Re}(z),\, \mathrm{Re}(w)\}}$ for $z, w \in \mathbb{C}$, which is a consequence of the Fundamental Theorem of Calculus for line integrals, we obtain that

$$\begin{aligned}
\big|(\mathcal{L}_{s\phi}^m \chi_\omega)(x) - (\mathcal{L}_{s\phi}^m \chi_\omega)(z)\big| &= \big|e^{sS_m\phi(y_x^\omega)} - e^{sS_m\phi(y_z^\omega)}\big| \\
&\leqslant |s|\, |S_m\phi(y_x^\omega) - S_m\phi(y_z^\omega)|\, e^{a \max\{\mathrm{Re}(S_m\phi(y_x^\omega)),\, \mathrm{Re}(S_m\phi(y_z^\omega))\}} \\
&\leqslant \frac{\|\phi\|_{C^{0,\beta}}}{\tau^{-\beta} - 1} |s|\, d_\tau(x,z)^\beta \sup_{y \in C_\omega} e^{aS_m\phi(y)}.
\end{aligned}$$

Since $|s| = \sqrt{a^2 + b^2} \leqslant |a| + |b| \leqslant a_0 + |b| \leqslant \big(1 + \frac{a_0}{b_0}\big)|\mathrm{Im}(s)|$, it follows that

$$\big|(\mathcal{L}_{s\phi}^m \chi_\omega)(x) - (\mathcal{L}_{s\phi}^m \chi_\omega)(z)\big| \leqslant \frac{\big(1 + \frac{a_0}{b_0}\big)\|\phi\|_{C^{0,\beta}}}{\tau^{-\beta} - 1} |\mathrm{Im}(s)|\, d_\tau(x,z)^\beta \sup_{y \in C_\omega} e^{aS_m\phi(y)}.$$

Combining the estimates from these three cases, we conclude that

$$|\mathcal{L}_{s\phi}^m \chi_\omega|_\beta \leqslant \max\bigg\{1,\, \frac{\big(1 + \frac{a_0}{b_0}\big)\|\phi\|_{C^{0,\beta}}}{\tau^{-\beta} - 1} |\mathrm{Im}(s)|\bigg\} \sup_{y \in C_\omega} e^{aS_m\phi(y)} \leqslant C_0 |\mathrm{Im}(s)| \sup_{y \in C_\omega} e^{aS_m\phi(y)},$$

where $C_0 \coloneqq \max\Big\{\frac{1}{b_0},\, \frac{\big(1 + \frac{a_0}{b_0}\big)\|\phi\|_{C^{0,\beta}}}{\tau^{-\beta} - 1}\Big\}$. Thus we have

$$\begin{aligned}
\|\mathcal{L}_{s\phi}^m \chi_\omega\|_{C^{0,\beta}} &= \|\mathcal{L}_{s\phi}^m \chi_\omega\|_\infty + |\mathcal{L}_{s\phi}^m \chi_\omega|_\beta \\
&\leqslant \big(1 + C_0|\mathrm{Im}(s)|\big) \sup_{y \in C_\omega} e^{aS_m\phi(y)} \\
&\leqslant C|\mathrm{Im}(s)| \sup_{y \in C_\omega} e^{aS_m\phi(y)},
\end{aligned}$$

where the constant $C \coloneqq \frac{1}{b_0} + C_0$ only depends on a_0, b_0, τ, β, and ϕ. Summing over all admissible words ω of length m yields

$$\sum_{|\omega| = m} \|\mathcal{L}_{s\phi}^m \chi_\omega\|_{C^{0,\beta}} \leqslant C|\mathrm{Im}(s)| \sum_{|\omega| = m} \sup_{y \in C_\omega} e^{aS_m\phi(y)}. \tag{A.4}$$

Denote

$$Z_m(a\phi) \coloneqq \sum_{|\omega| = m} \sup_{y \in C_\omega} e^{aS_m\phi(y)}.$$

It is a classical result that $\lim_{m \to +\infty} \frac{1}{m} \log Z_m(a\phi) = P(\sigma, a\phi)$ (see e.g. [Wal82, Theorem 9.6]). We now show that this convergence is uniform for $a \in [-a_0, a_0]$. Indeed, a straightforward calculation shows that for each $m \in \mathbb{N}$, the function $a \mapsto p_m(a) \coloneqq \frac{1}{m} \log Z_m(a\phi)$ is Lipschitz continuous on $\mathbb{R}$ with a Lipschitz constant $\|\phi\|_\infty$, and is bounded on $[-a_0, a_0]$ by $a_0\|\phi\|_\infty + \log(\mathrm{card}(S))$. This implies that the sequence of functions $\{p_m\}_{m\in\mathbb{N}}$ is equicontinuous and uniformly bounded on $[-a_0, a_0]$. By the Arzelà–Ascoli theorem, every subsequence $\{p_{m_k}\}_{k\in\mathbb{N}}$, which is equicontinuous and uniformly bounded on $[-a_0, a_0]$, has a further subsequence $\{p_{m_{k_j}}\}_{j\in\mathbb{N}}$ that converges uniformly on $[-a_0, a_0]$. Note that such a subsequential limit must be $P(\sigma, \cdot\,\phi)$ since $\lim_{m\to+\infty} p_m(a) = P(\sigma, a\phi)$. A standard result in topology states that if every subsequence of a sequence in a topological space has a further subsequence converging to the same point, then the original sequence itself converges to that point. Therefore, the convergence $\lim_{m\to+\infty} p_m(a) = P(\sigma, a\phi)$ is uniform in $a \in [-a_0, a_0]$. Hence, there exists a constant $K_\varepsilon > 0$ such that for all $m \in \mathbb{N}$ and $a \in [-a_0, a_0]$, we have

$$Z_m(a\phi) \leqslant K_\varepsilon\, e^{m(P(\sigma,a\phi)+\varepsilon)}, \tag{A.5}$$

and (A.1) follows immediately from (A.4) and (A.5) by setting $C_\varepsilon \coloneqq \tau^{-\beta} C K_\varepsilon$.

Finally, by substituting the estimate (A.1) into (A.3), we obtain

$$\begin{aligned} |Z_n(s) - T_n(s)| &\leqslant \tau^{-\beta} \sum_{m=2}^{n} \tau^{\beta m} \left\|\mathcal{L}_{s\phi}^{n-m}\right\|_{C^{0,\beta}} \sum_{|\omega|=m} \|\mathcal{L}_{s\phi}^{m} \chi_\omega\|_{C^{0,\beta}} \\ &\leqslant C_\varepsilon |\mathrm{Im}(s)| \sum_{m=2}^{n} \left\|\mathcal{L}_{s\phi}^{n-m}\right\|_{C^{0,\beta}} \tau^{\beta m} e^{m(P(\sigma,a\phi)+\varepsilon)}, \end{aligned}$$

establishing (A.2). □

The following estimate for the partition function is an immediate consequence of Lemma A.2.

Corollary A.4. *For all $a_0 > 0$, $b_0 > 0$, $\varepsilon > 0$, $n \in \mathbb{N}$, and $s \in \mathbb{C}$ satisfying $|\mathrm{Re}(s)| \leqslant a_0$ and $|\mathrm{Im}(s)| \geqslant b_0$, we have*

$$\left|\sum_{\sigma^n x = x} e^{sS_n\phi(x)}\right| \leqslant C_\varepsilon |\mathrm{Im}(s)| \sum_{m=1}^{n} \left\|\mathcal{L}_{s\phi}^{n-m}\right\|_{C^{0,\beta}} \left(\tau^\beta e^{P(\sigma,\mathrm{Re}(s)\phi)+\varepsilon}\right)^m, \tag{A.6}$$

where C_ε is the constant from Lemma A.2, which depends only on ε, a_0, b_0, τ, β, ϕ, and the ambient parameters of the subshift $\left(\Sigma_A^+, \sigma_A\right)$.

Proof. When $n = 1$, (A.6) follows immediately from Lemma A.3 and (A.1) in Lemma A.2.

We now consider the case $n \geqslant 2$. Let $x_j \in C_j$ be an arbitrary point for each $j \in S$. By (A.1), we have

$$\begin{aligned} I \coloneqq \left|\sum_{j\in S} \mathcal{L}_{s\phi}^{n} \chi_j(x_j)\right| &\leqslant \sum_{j\in S} \left|\mathcal{L}_{s\phi}^{n-1}(\mathcal{L}_{s\phi}\chi_j)(x_j)\right| \leqslant \left\|\mathcal{L}_{s\phi}^{n-1}\right\|_{C^{0,\beta}} \sum_{j\in S} \left\|\mathcal{L}_{s\phi}\chi_j\right\|_{C^{0,\beta}} \\ &\leqslant \left\|\mathcal{L}_{s\phi}^{n-1}\right\|_{C^{0,\beta}} C_\varepsilon |\mathrm{Im}(s)| \tau^\beta e^{P(\sigma,\mathrm{Re}(s)\phi)+\varepsilon}, \end{aligned}$$

where C_ε is the constant from Lemma A.2. Hence, using the triangle inequality and (A.2) in Lemma A.2, we deduce that

$$\begin{aligned} \left|\sum_{\sigma^n x = x} e^{sS_n\phi(x)}\right| &\leqslant I + \left|\sum_{\sigma^n x = x} e^{sS_n\phi(x)} - \sum_{j\in S} \mathcal{L}_{s\phi}^{n} \chi_j(x_j)\right| \\ &\leqslant C_\varepsilon |\mathrm{Im}(s)| \left\|\mathcal{L}_{s\phi}^{n-1}\right\|_{C^{0,\beta}} \tau^\beta e^{P(\sigma,\mathrm{Re}(s)\phi)+\varepsilon} + C_\varepsilon |\mathrm{Im}(s)| \sum_{m=2}^{n} \left\|\mathcal{L}_{s\phi}^{n-m}\right\|_{C^{0,\beta}} \left(\tau^\beta e^{P(\sigma,\mathrm{Re}(s)\phi)+\varepsilon}\right)^m \\ &= C_\varepsilon |\mathrm{Im}(s)| \sum_{m=1}^{n} \left\|\mathcal{L}_{s\phi}^{n-m}\right\|_{C^{0,\beta}} \left(\tau^\beta e^{P(\sigma,\mathrm{Re}(s)\phi)+\varepsilon}\right)^m. \end{aligned}$$

This establishes (A.6). □

References

[Bal00] Baladi, V., *Positive transfer operators and decay of correlations*, vol. 16, World scientific, Singapore, 2000.

[BM17] Bonk, M. and Meyer, D., *Expanding Thurston Maps*, Amer. Math. Soc., Providence, RI, 2017.

[Bow75] Bowen, R., *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Lecture Notes in Mathematics, Springer, Berlin, 1975.

[DPTUZ21] Das, T., Przytycki, F., Tiozzo, G., Urbański, M., and Zdunik, A., Thermodynamic formalism for coarse expanding dynamical systems, *Comm. Math. Phys.* 384 (2021), 165–199.

[Hei01] Heinonen, J., *Lectures on Analysis on Metric Spaces*, Springer, New York, 2001.

[Kat95] Kato, T., *Perturbation theory for linear operators*, Classics in Mathematics, Reprint of the 1980 edition, Springer-Verlag, Berlin, 1995, pp. xxii+619.

[KH95] Katok, A. and Hasselblatt, B., *Introduction to the Modern Theory of Dynamical Systems*, Cambridge Univ. Press, Cambridge, 1995.

[Kit98] Kitchens, B. P., *Symbolic dynamics: one-sided, two-sided and countable state Markov shifts*, Springer, Berlin, 1998.

[Li16] Li, Z., Periodic points and the measure of maximal entropy of an expanding Thurston map, *Trans. Amer. Math. Soc.* 368 (2016), 8955–8999.

[Li17] Li, Z., *Ergodic theory of expanding Thurston maps*, Atlantis Press, Paris, 2017.

[Li18] Li, Z., Equilibrium states for expanding Thurston maps, *Comm. Math. Phys.* 357 (2018), 811–872.

[LZ24a] Li, Z. and Zheng, T., Prime orbit theorems for expanding Thurston maps: Dirichlet series and orbifolds, *Adv. Math.* 443 (2024), 109600.

[LZ24b] Li, Z. and Zheng, T., Prime orbit theorems for expanding Thurston maps: Genericity of strong non-integrability condition, *Adv. Math.* 450 (2024), 109765.

[LZ24c] Li, Z. and Zheng, T., Prime orbit theorems for expanding Thurston maps: Lattès maps and split Ruelle operators, *Adv. Math.* 449 (2024), 109723.

[Nau05] Naud, F., Expanding maps on Cantor sets and analytic continuation of zeta functions, *Ann. Sci. École Norm. Sup. (4)* 38 (2005), 116–153.

[PP90] Parry, W. and Pollicott, M., Zeta functions and the periodic orbit structure of hyperbolic dynamics, *Astérisque* 187 (1990), 1–268.

[PS12] Petkov, V. and Stoyanov, L., Distribution of periods of closed trajectories in exponentially shrinking intervals, *Comm. Math. Phys.* 310 (2012), 675–704.

[Pol84] Pollicott, M., A complex Ruelle-Perron-Frobenius theorem and two counterexamples, *Ergodic Theory Dynam. Systems* 4 (1984), 135–146.

[PS98] Pollicott, M. and Sharp, R., Exponential error terms for growth functions on negatively curved surfaces, *Amer. J. Math.* 120 (1998), 1019–1042.

[PS01] Pollicott, M. and Sharp, R., Error terms for closed orbits of hyperbolic flows, *Ergodic Theory Dynam. Systems* 21 (2001), 545–562.

[PU10] Przytycki, F. and Urbański, M., *Conformal Fractals: Ergodic Theory Methods*, Cambridge Univ. Press, Cambridge, 2010.

[Rue90] Ruelle, D., An extension of the theory of Fredholm determinants, *Inst. Hautes Études Sci. Publ. Math.* (1990), 175–193.

[SS22] Sharp, R. and Stylianou, A., Statistics of multipliers for hyperbolic rational maps, *Discrete Contin. Dyn. Syst.* 42 (2022), 1225–1241.

[Wal82] Walters, P., *An Introduction to Ergodic Theory*, Springer, New York, 1982.

ZHIQIANG LI, SCHOOL OF MATHEMATICAL SCIENCES & BEIJING INTERNATIONAL CENTER FOR MATHEMATICAL RESEARCH, PEKING UNIVERSITY, BEIJING 100871, CHINA

Email address: `zli@math.pku.edu.cn`

Xianghui Shi, Beijing International Center for Mathematical Research, Peking University, Beijing 100871, China

Email address: xhshi@pku.edu.cn