COMPUTABLE THERMODYNAMIC FORMALISM

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ABSTRACT. We investigate the problem of constructing approximations for equilibrium states of a general family of computable dynamical systems. Specifically, we devise two complementary general approaches to study the computability of equilibrium states for non-uniformly expanding systems. The first approach applies to dynamical systems whose topological pressure functions have effective approximations and whose measure-theoretic entropy functions are upper semi-continuous. As an application, we establish the computability of the equilibrium states for Misiurewicz–Thurston rational maps (i.e., postcritically-finite rational maps without periodic critical points) with Hölder continuous potentials. The second approach involves prescribed Jacobians of equilibrium states. Using this approach, we show the computability of the unique measure of maximal entropy for an expanding Thurston map derived from the barycentric subdivisions, even though its measure-theoretic entropy function is not upper semi-continuous.

CONTENTS

1. INTRODUCTION

1.1. **Background and motivation.** In recent years, automated computations have achieved remarkable success in the study of various natural and social phenomena modeled by dynamical systems. Consequently, the theory of dynamical systems has attracted significant interest from computer scientists (see a recent survey by Yampolsky [Yam21]). There has been a dramatic growth in research on the computability and computational complexity of dynamical invariants, such as topological entropy and pressure (see for example, [Spa07, HM10, GHRS20, BDWY22]). Regarding complex dynamics, a groundbreaking work by Braverman and Yampolsky investigates the computability and complexity of Julia sets [BY06, BY09].

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They show that accurate computations of these intriguing images of Julia sets are constrained by certain limitations. For further research on algorithmic aspects of Julia sets, we refer the reader to some recent works by Rojas and Yampolsky [RY21] and Dudko and Yampolsky [DY21] and references therein.

The sensitivity to initial conditions and the inherent instability of many important physical systems imply that the observations on a computer screen may be entirely unrelated to the intended simulations. In the introduction of [BBRY11], a modern paradigm of numerical study of chaos is mentioned: while the simulation of an individual orbit for an extended period of time does not make practical sense, one could study the limit set of a typical orbit (both as a spatial object and as a statistical distribution). On the statistical side, researchers often focus on computing invariant measures or estimating their properties to understand the system's overall behavior. Additionally, there are studies on the algorithmic aspects of important invariant measures (see for example, [GHR11, BBRY11] and references therein), as well as on other geometrically and statistically important measures such as harmonic measures [BGRY22].

In general, a dynamical system may possess uncountably many invariant measures (typically forming an infinite-dimensional set). Consequently, the majority of these measures will not be algorithmically describable. Nevertheless, this issue is not particularly problematic, as we shall concentrate on those invariant measures that possess the most significant mathematical or physical relevance.

Thermodynamic formalism serves as a potent tool for creating and studying invariant measures with prescribed local behavior under iterations of dynamical systems. This theory, which draws inspiration from statistical mechanics, was pioneered by Ruelle, Sinai, Bowen, and others in the 1970s ([Dob68, Sin72, Bow75, Wal82]). Since its birth, thermodynamic formalism has been extensively applied in various classical contexts (see for example, [Rue89, Prz90, KH95, Zin96, BS03, Oli03, Yur03]) and has remained at the frontier and core of research in dynamical systems.

The concepts of measure-theoretic entropy and topological entropy in dynamical systems have their roots in the early works on the notions of entropy by Boltzmann and Gibbs (statistical mechanics 1875), von Neumann (quantum mechanics 1932), and Shannon (information theory 1948). These notions of entropy are all designed to describe the complexity of their respective systems or objects. Furthermore, in recent years, there have been many exciting developments and diverse applications of entropy and complexity theory. We refer the reader to Braverman's report at the International Congress of Mathematicians (ICM) in 2022 [Bra23].

Specifically, for a continuous map defined on a compact metric space, we can consider the *topological pressure* as a weighted version of the *topological entropy*, with the weight induced by a real-valued continuous function, called a *potential*. The Variational Principle identifies the topological pressure with the supremum of its measure-theoretic counterpart, the *measure-theoretic pressure* [Bow75, Wal75], over all invariant Borel probability measures. Notably, for the constant potential with value zero, the measure-theoretic pressure reduces to the *measure-theoretic entropy*, which describes the complexity of a dynamical system from the perspective of measures. The central focus of thermodynamic formalism is on invariant measures called *equilibrium states*, which maximize the measure-theoretic pressure. In particular, for a constant potential, an equilibrium state reduces to a *measure of maximal entropy*. In many settings, equilibrium states also describe the weighted distribution of iterated preimages and periodic orbits (see for example, [Li18, LS24, BD23, BD24]) and random backward orbits (see for example, [HT03, Li18]).

For instance, in the realm of complex dynamics, Brolin–Lyubich measures [Bro65, Lyu82] represent the measures of maximal entropy for rational maps. In [BBRY11], a uniform algorithm was devised to compute the Brolin–Lyubich measures. This work complements the discovery of polynomials with computable coefficients but non-computable Julia sets, as explored in the pioneering works of Braverman and Yampolsky [BY06, BY09], which can be traced back to a question posed by Milnor (see [BY06, Section 1]).

The computability of Brolin–Lyubich measures poses a paradox. Intuitively, one might assume that a measure contains more information than its support. However, computable analysis demonstrates that there exists a computable invariant probability measure whose support is, in fact, non-computable. Indeed, this paradox can be reconciled by interpreting these two results as the computability properties of the same physical objects from geometric and statistical perspectives, respectively. Under this statistical perspective, questions about the computability of other important measures, such as equilibrium states, become particularly interesting.

In dynamical systems, uniformly expanding systems are generally regarded as relatively straightforward to analyze. However, without any assumptions regarding expansion, the analysis of the system becomes infeasible. Therefore, relaxing the traditional assumptions of expansion is considered a Holy Grail in this field, which is the most challenging and rewarding aspect of research in dynamical systems.

In this paper, building on the work of [BBRY11] and [BHLZ24], we study computability questions for equilibrium states in dynamical systems far beyond the uniformly expanding ones.

We propose two approaches to establish the computability of equilibrium states, each with its own strengths and weaknesses. These approaches are complementary, aiming to be applicable to a wide variety of dynamical systems where thermodynamic formalism has been extensively studied. The first approach is suitable for dynamical systems whose measure-theoretic entropy functions are upper semi-continuous and whose topological pressure functions can be effectively approximated. For dynamical systems whose measure-theoretic entropy functions may lack upper semi-continuity, we implement the second approach, which establishes the computability of equilibrium states by considering the prescribed Jacobians for equilibrium states.

Furthermore, we focus on expanding Thurston maps as case studies to illustrate our approaches. Recall that a Thurston map is a non-homeomorphic branched covering map on a topological 2-sphere S^2 such that each of its critical points has a finite orbit (called *postcritically-finite*). The most important examples are given by postcritically-finite rational maps on the Riemann sphere $\widehat{\mathbb{C}}$.

Thurston maps play a central role in the study of complex dynamics, and there has been active research on the algorithmic aspects of these maps. For example, the work of Bonnet, Braverman, and Yampolsky [BBY12] devised an algorithm to determine whether a Thurston map is Thurston equivalent to a rational map. If the rational map is a quadratic rational map, the paper of Hubbard and Schleicher [HS94] provides an algorithm that, given a convenient description of the Thurston map, outputs the coefficients of the rational map. Finally, on the decidability of Thurston equivalence, we refer the reader to some recent works of Selinger, Rafi, and Yampolsky [SY15, RSY20].

Inspired by Sullivan's dictionary and their interest in Cannon's Conjecture [Can94], Bonk and Meyer [BM10, BM17], as well as Haïssinsky and Pilgrim [HP09], studied a subclass of Thurston maps, called *expanding Thurston maps*, by imposing some additional condition of weak expansion (see Definition 6.2). Furthermore, ergodic theory for expanding Thurston maps has been thoroughly investigated in [BM10, BM17, HP09, Li15, Li17, Li18, LS24]. Notably, in [Li18], for expanding Thurston maps and Hölder continuous potentials, the third-named author of the current paper developed the thermodynamic formalism and investigates the existence, uniqueness, and ergodic properties of equilibrium states. We remark that expanding Thurston maps are not expansive and not even h-expansive, and the ones with a periodic critical point cannot even be asymptotically h-expansive [Li15].

Recently, in [LS24], for an expanding Thurston map, the third-named and fourth-named authors of the current paper demonstrated that the measure-theoretic entropy function is upper semi-continuous if and only if the map has no periodic critical points. Based on the aforementioned research, we employ our methods to investigate the computability of equilibrium states for expanding Thurston maps to showcase the use of our two distinct approaches.

1.2. **Main results.** Our main results include two approaches (Theorems 1.1 and 1.3) for establishing the computability of equilibrium states for certain computable systems, as well as two corresponding applications (Theorems 1.2 and 1.4) to equilibrium states for expanding Thurston maps. Now we introduce the first approach.

Theorem 1.1. *Let $(X, \rho, \mathcal{S})$ be a computable metric space, X a recursively compact set, and $T\colon X \to X$ a computable function. Assume that $\phi\colon X \to \mathbb{R}$ is a computable function that satisfies the following properties:*

(i) *There exists a sequence $D = \{\psi_i\}_{i\in\mathbb{N}}$ of uniformly computable functions on X such that the closure $\overline{D}$ of D in $C(X)$ contains a neighborhood of ϕ and there exists an algorithm that, on input $i \in \mathbb{N}$, outputs a non-increasing sequence of real values converging to the topological pressure $P(T, \psi_i)$.*

(ii) *$P(T,\phi)$ is lower semi-computable.*

Then $C(X)^*_{\phi,P(T,\cdot)}$ is a recursively compact subset of $\mathcal{P}(X)$. Additionally, if the measure-theoretic entropy map $\nu \mapsto h_\nu(T)$ is upper semi-continuous on $\mathcal{M}(X,T)$, and $\mathcal{E}(T,\phi)=\{\mu_\phi\}$, then the equilibrium state μ_ϕ is a computable measure.

Here $C(X)^*_{\phi,P(T,\cdot)}$ is defined in Definition 4.1 and $\mathcal{E}(T,\phi)$ denotes the set of all the equilibrium states for the map T and the potential ϕ. For more details on computable measure theory, we refer the reader to Subsection 3.1.3.

As an application, the computability of the unique equilibrium state for a Misiurewicz–Thurston rational map and a Hölder potential is established as follows.

Theorem 1.2. *Let $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a Misiurewicz–Thurston rational map (i.e., a postcritically-finite rational map without periodic critical points), σ be the chordal metric, and $\alpha \in (0,1]$. Then there exists an algorithm that satisfies the following property:*

For each real-valued Hölder continuous function $\phi \in C^{0,\alpha}(\widehat{\mathbb{C}},\sigma)$, this algorithm outputs a rational linear combination of finite Dirac measures which are supported on some points in $\mathbb{Q}(\widehat{\mathbb{C}})$ as a 2^{-n}-approximation in the Wasserstein–Kantorovich metric W_σ for the unique equilibrium state μ_ϕ for the map f and the potential ϕ, after inputting the following data in this algorithm:

(i) *an algorithm computing the potential ϕ,*
(ii) *an algorithm computing all the coefficients of the rational map f,*
(iii) *a rational constant Q such that $|\phi|_{\alpha,\sigma} \leqslant Q$, and*
(iv) *a constant $n \in \mathbb{N}$.*

Here $\mathbb{Q}(\widehat{\mathbb{C}})$ denotes the set $\{a+b\mathbf{i} : a,\, b \in \mathbb{Q}\} \cup \{\infty\}$.

It is worth noting that Theorem 1.1 can be applied to establish the computability of equilibrium states for a wide range of dynamical systems with a unique equilibrium state and an upper semi-continuous measure-theoretic entropy function (for example, rational maps with Hölder continuous hyperbolic potential). Due to space limitations, we focus on the current examples and postpone further investigations to future works.

However, because of the diversity of dynamical systems, measure-theoretic entropy functions are not always upper semi-continuous. To overcome this issue, we establish a second approach by considering the prescribed Jacobians for equilibrium states.

Theorem 1.3. *Let $(X,\, \rho,\, \mathcal{S})$ be a computable metric space, X a recursively compact set, $\phi\colon X \to \mathbb{R}$ a computable function, and $T\colon X \to X$ a finite-to-one and computable map with finite topological entropy and finitely many singular points. Assume that C is a recursively compact subset of X and $\{U_j\}_{j\in\mathbb{N}}$ is a sequence of uniformly lower semi-computable open sets of X satisfying the following properties:*

(a) *$C \smallsetminus \operatorname{Sing}(T) = \bigcup_{j\in\mathbb{N}} U_j$.*
(b) *T is injective and open on U_j for each $j \in \mathbb{N}$.*

Suppose that $J\colon X \to \mathbb{R}$ is a Borel measurable positive function that is upper semi-computable on X and satisfies the following properties:

(i) *For each $x \in T(C \smallsetminus \operatorname{Sing}(T))$,*
$$\sum_{y\in T^{-1}(x)\cap(C\smallsetminus \operatorname{Sing}(T))} \frac{1}{J(y)} = 1.$$
(ii) *There exists a Borel measurable function $h\colon X \to \mathbb{R}$ satisfying that*
$$J(x) = \exp(P(T,\phi) - \phi(x) + h(T(x)) - h(x)) \quad \text{for each } x \in C \smallsetminus \operatorname{Sing}(T).$$

If $\mathcal{E}(T,\phi) \cap \mathcal{P}(X, C\smallsetminus \operatorname{Sing}(T))$ is weak* compact, then $\mathcal{E}(T,\phi)\cap\mathcal{P}(X,C)$ and $\mathcal{E}(T,\phi)\cap\mathcal{P}(X,C\smallsetminus \operatorname{Sing}(T))$ are both recursively compact subsets of $\mathcal{P}(X)$. In particular, if $\mathcal{E}(T,\phi)\cap\mathcal{P}(X,C\smallsetminus \operatorname{Sing}(T)) = \{\mu_\phi\}$, then the equilibrium state μ_ϕ is a computable measure.

Here singular points (for T) are defined in Definition 5.2 and $\mathrm{Sing}(T)$ is the set of all the singular points for T. For each Borel subset $C \in \mathcal{B}(X)$, we denote by $\mathcal{P}(X, C)$ the set $\{\mu \in \mathcal{P}(X) : \mu(C) = 1\}$. For more details on computable measure theory, we refer the reader to Subsection 3.1.3.

Theorem 1.3 generalizes the methods in [BBRY11, Theorem A] and [BHLZ24, Theorem 1.1]. With Theorem 1.3, we can focus on equilibrium states supported on certain subsets. Moreover, to address the shortcomings of Theorem 1.1, we can apply Theorem 1.3 to some dynamical systems whose measure-theoretic entropy functions are not upper semi-continuous, such as some partially hyperbolic systems and expanding Thurston maps with periodic critical points.

To showcase the use of Theorem 1.3 and limit unnecessary technicalities, we apply Theorem 1.3 to establish the computability of the unique measure of maximal entropy τ for some expanding Thurston map g from the barycentric subdivisions (see Subsection 6.3) and obtain the following result.

Theorem 1.4. *In the computable metric space $(S_\triangle, d_\triangle, \mathbb{Q}(S_\triangle))$, there exists a unique measure of maximal entropy μ for the map g from the barycentric subdivisions and μ is a computable measure.*

Similarly, we can apply Theorem 1.3 to extend the above result to more general expanding Thurston maps and potentials. We omit such applications in the current work.

1.3. **Strategy and organization of the paper.** In Section 2, we fix some notation that will be used throughout the paper. In Section 3, we introduce some basic notions and results in computable analysis (Subsection 3.1) and thermodynamic formalism (Subsection 3.2).

Next, we establish Theorems 1.1 and 1.3 in Sections 4 and 5, respectively. Indeed, the proofs of Theorems 1.1 and 1.3 follow the same general philosophy: we first establish the recursive compactness of some subset K of the set $\mathcal{E}(T, \phi)$ of all the equilibrium states for the map T and the potential ϕ. Consequently, by Proposition 3.18 (i), the additional assumption that subset $K = \{\mu\}$ is a singleton set implies that equilibrium state μ is computable.

In Section 4, we establish Theorem 1.1. First, we recall some notations and results in functional analysis. Among them, Lemma 4.3 gives a bijection mapping the tangent space $C(X)^*_{\phi,P(T,\cdot)}$ (see Definition 4.1) to the set $\mathcal{M}(X, T)$ of T-invariant measures on X, which allows us to see $C(X)^*_{\phi,P(T,\cdot)}$ as a subset of $\mathcal{M}(X, T)$. In this regard, the weak* compactness of $C(X)^*_{\phi,P(T,\cdot)}$ follows from Lemma 4.3. However, the set $\mathcal{E}(T, \phi)$ of all the equilibrium states for the map T and the potential ϕ may not be weak* compact. Note that recursively compact sets in $\mathcal{P}(X)$ are always weak* compact. Therefore, in the proof of Theorem 1.1, instead of establishing the recursive compactness of $\mathcal{E}(T, \phi)$ directly, we first establish the recursive compactness of $C(X)^*_{\phi,P(T,\cdot)}$. Then by Corollary 4.4, for a dynamical system whose measure-theoretic entropy function is upper semi-continuous, we obtain that $\mathcal{E}(T, \phi) = C(X)^*_{\phi,P(T,\cdot)}$. Consequently, $\mathcal{E}(T, \phi)$ is recursively compact.

In Section 5, we present another approach to studying the computability of equilibrium states. In Subsection 5.1, we review the definitions of Jacobians, singular points, and Rokhlin's formula. Then for each finite-to-one continuous map T with finitely many singular points and each T-invariant Borel probability measure μ, we define the corresponding transfer operator and investigate its properties in Proposition 5.4. Using the transfer operators and their properties, we establish Theorem 5.5 to give an equivalent description of the Jacobians of invariant measures. Thus, by the Variational Principle, Corollary 5.6 follows, allowing us to find equilibrium states by checking if a Jacobian for T equals the desired function J.

Next, in Subsection 5.2, we generalize the methods in [BBRY11, BHLZ24] to establish Theorem 1.3. More precisely, for each $\mu \in \mathcal{M}(X, T)$, we check if a Jacobian for T with respect to μ equals the desired function J to determine if μ is an equilibrium state. However, compared to the uniformly expanding systems in [BBRY11, BHLZ24], the system in Theorem 1.1 admits finitely many singular points, which presents the most challenging task we need to overcome in the proof. Indeed, by the definition of singular points, for each $x \in \mathrm{Sing}(T)$, there exists no neighborhood U of x on which T is injective and open. Therefore, we cannot use the method in [BBRY11, BHLZ24] to compute the values of the Jacobians near the singular points. To overcome this challenge, we follow three main steps.

First, we apply a similar method from [BBRY11, BHLZ24] to check if a Jacobian for T with respect to μ equals the desired function J in the "good" part $X \setminus \operatorname{Sing}(T)$. More precisely, we establish Claims 1 and 2 to show that the set Ψ of invariant measures whose Jacobians are smaller than the given function J almost everywhere is a recursively compact set. Then by Claim 2 and Corollary 5.6, Claim 3 follows and shows that $\Psi \cap \mathcal{P}(X, C \setminus \operatorname{Sing}(T)) = \mathcal{E}(T, \phi) \cap \mathcal{P}(X, C \setminus \operatorname{Sing}(T))$.

The second step is to consider the general equilibrium states μ with $\mu(\operatorname{Sing}(T)) > 0$, i.e., to determine the relation between the sets Ψ and $\Psi \cap \mathcal{P}(X, C \setminus \operatorname{Sing}(T))$. Note that for each T-invariant measure μ, $\mu(\{x\}) > 0$ implies that x is a periodic point for T with period $N \in \mathbb{N}$. Then μ can be expressed as a linear combination of the measure $\frac{1}{N}\sum_{i=0}^{N-1}\delta_{T^i(x)}$ and a measure μ' satisfying that $\mu'(\{x\}) = 0$. Hence, Claim 4 follows and implies that every measure in Ψ can be expressed as a linear combination of one measure in $\mathcal{M}^* \cap \Psi$ and one measure in $\mathcal{P}(X, C \setminus \operatorname{Sing}(T)) \cap \Psi$. Here, $\mathcal{M}^*$ denotes the set of invariant measures supported on the union of all periodic orbits containing singular points.

Finally, we shall establish the recursive compactness of the sets $\mathcal{E}(T, \phi) \cap \mathcal{P}(X, C \setminus \operatorname{Sing}(T))$ and $\mathcal{E}(T, \phi) \cap \mathcal{P}(X, C)$ from the recursive compactness of the set Ψ. To this end, we establish Proposition 5.8 at the beginning of Subsection 5.2 to show the equivalence of the recursive compactness of the sets K and $\operatorname{cl}(K, \{\mu_0\})$ under some additional assumptions. Here, $\operatorname{cl}(K, \{\mu_0\}) \coloneqq \{\lambda a + (1-\lambda)\mu_0 : \lambda \in [0,1],\, a \in K\}$. As an immediate corollary, we establish Corollary 5.9, which extends the singletons to some convex hulls with finitely many vertices. With this corollary and Claims 3 and 4, we complete the proof of Theorem 1.3.

We would like to add a remark on these two approaches. In thermodynamic formalism, we can verify whether the topological pressure is equal to the sum of the measure-theoretic entropy and the integral of the potential function to identify equilibrium states. It is worth noting that computing the topological pressure function and the integral of the potential is straightforward, and the challenging aspect lies in calculating the measure-theoretic entropy function $\mu \mapsto h_\mu(T)$. Our two approaches correspond to two different ways to deal with this problem. In Section 4, we assume that $h_\mu(T) = \overline{h}_\mu(T)$ for each $\mu \in \mathcal{M}(X,T)$ and apply Lemma 4.5 to compute it. In Section 5, Rokhlin's formula inspires us to consider Jacobians for equilibrium states.

In Section 6, we end this paper with two applications. We first recall expanding Thurston maps in Subsection 6.1. Then we apply Theorem 1.1 to establish Theorem 1.2 in Subsection 6.2 and apply Theorem 1.3 to establish Theorem 1.4 in Subsection 6.3.

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2. Notation

Let $\mathbb{C}$ be the complex plane and $\widehat{\mathbb{C}}$ be the Riemann sphere. Let $\mathbf{i}$ denote the imaginary unit in the complex plane $\mathbb{C}$. Define the chordal metric σ on $\widehat{\mathbb{C}}$ as follows: $\sigma(z,w) \coloneqq \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}$ for all $z,\, w \in \mathbb{C}$, and $\sigma(\infty, z) = \sigma(z, \infty) \coloneqq \frac{2}{\sqrt{1+|z|^2}}$ for all $z \in \mathbb{C}$. Let S^2 denote an oriented topological 2-sphere. We use $\mathbb{N}$ to denote the set of integers greater than or equal to 1 and write $\mathbb{N}_0 \coloneqq \{0\} \cup \mathbb{N}$. We denote by $\mathbb{Q}^+$ (resp. $\mathbb{R}^+$) the set of all positive rational numbers (resp. positive real numbers). Moreover, denote the set of all finite subsets of $\mathbb{N}$ by $\mathbb{N}^*$. The symbol log denotes the logarithm to the base e. For $x \in \mathbb{R}$, we define $\lfloor x \rfloor$ as the greatest integer $\leqslant x$, $\lceil x \rceil$ the smallest integer $\geqslant x$, and $|x|^+ \coloneqq \max\{x, 0\}$. We denote by $\operatorname{sgn}(x)$ the sign function for each $x \in \mathbb{R}$. The cardinality of a set A is denoted by $\operatorname{card}(A)$.

Consider a map $f\colon X \to X$ on a set X. The inverse map of f is denoted by f^{-1}. We write f^n for the n-th iterate of f, and $f^{-n} \coloneqq (f^n)^{-1}$, for each $n \in \mathbb{N}$. We set $f^0 \coloneqq \mathrm{id}_X$, the identity map on X. For a real-valued function $\varphi\colon X \to \mathbb{R}$, we write $S_n\varphi(x) = S_n^f\varphi(x) \coloneqq \sum_{j=0}^{n-1}\varphi\bigl(f^j(x)\bigr)$. for each $x \in X$ and each $n \in \mathbb{N}_0$. We omit the superscript f when the map f is clear from the context. Note that when $n = 0$, by definition we always have $S_0\varphi = 0$.

Let (X, d) be a metric space. We denote by $\mathcal{B}(X)$ the σ-algebra of all Borel sets on X. For each subset $Y \subseteq X$, we denote the diameter of Y by $\operatorname{diam}_d(Y) \coloneqq \sup\{d(x,y) : x,\, y \in Y\}$, the interior of Y by $\operatorname{int}(Y)$, and the characteristic function of Y by $\mathbb{1}_Y$ which maps each $x \in Y$ to $1 \in \mathbb{R}$ and vanishes otherwise.

For each $r > 0$ and each $x \in X$, we denote the open (resp. closed) ball of radius r centered at x by $B_d(x, r)$ (resp. $\overline{B_d}(x, r)$). For each $r > 0$ and each set $K \subseteq X$, we denote the set $\{x \in X : d(x, K) < r\}$ by $B_d(K, r)$. We often omit the metric d in the subscript when it is clear from the context.

For a compact metric space (X, d) and a continuous map $g\colon X \to X$, we denote by $C(X)$ (resp. $B(X)$) the space of continuous (resp. bounded Borel) functions from X to $\mathbb{R}$, by $\mathcal{M}(X)$ (resp. $\mathcal{M}(X, g)$) the set of finite signed Borel measures (resp. g-invariant Borel probability measures) on X, and $\mathcal{P}(X)$ the set of Borel probability measures on X. Moreover, for each Borel subset $C \in \mathcal{B}(X)$, $\mathcal{P}(X, C)$ denotes the set $\{\mu \in \mathcal{P}(X) : \mu(C) = 1\}$. By the Riesz representation theorem (see for example, [Fol13, Theorems 7.8 and 7.17]), we identify the dual of $C(X)$ with the space $\mathcal{M}(X)$. For $\mu \in \mathcal{M}(X)$, we use $\|\mu\|$ to denote the total variation norm of μ, $\operatorname{supp} \mu$ the support of μ (the smallest closed set $A \subseteq X$ such that $|\mu|(X \setminus A) = 0$), and

$$\langle \mu, u \rangle \coloneqq \int u \,\mathrm{d}\mu$$

for each $u \in C(X)$. If we do not specify otherwise, we equip $C(X)$ with the uniform norm $\|\cdot\|_{C(X)} \coloneqq \|\cdot\|_\infty$, and equip $\mathcal{M}(X)$, $\mathcal{P}(X)$, and $\mathcal{M}(X, g)$ with the weak* topology.

The space of real-valued Hölder continuous functions with an exponent $\alpha \in (0, 1]$ on a compact metric space (X, d) is denoted as $C^{0,\alpha}(X, d)$. For each $\phi \in C^{0,\alpha}(X, d)$,

$$|\phi|_{\alpha,d} \coloneqq \sup\left\{ \frac{|\phi(x) - \phi(y)|}{d(x, y)^\alpha} : x,\, y \in X,\, x \neq y \right\}.$$

For a compact metric space (X, d), the *Wasserstein–Kantorovich metric* W_d on $\mathcal{P}(X)$ is defined by

$$W_d(\mu, \nu) \coloneqq \sup\bigl\{ |\langle \mu, f \rangle - \langle \nu, f \rangle| : f \in C^{0,1}(X, d), |f|_{1,d} \leqslant 1 \bigr\}. \tag{2.1}$$

Moreover, for Borel probability measures in $\mathcal{P}(X)$, the convergence in the Wasserstein–Kantorovich metric W_d is equivalent to the convergence in the weak* topology (see for example, [Vil09, Corollary 6.13]).

3. Preliminaries

3.1. **Computable Analysis.** In this subsection, we recall some notions and results from Computable Analysis. The definitions we adopt are consistent with those in [Wei00]. Consequently, it is convenient to consider the algorithms or machines mentioned below as Type-2 machines, as defined in [Wei00, Definition 2.1.1]. For more details, we refer the reader to [BBRY11, Section 3], [GHR11, Section 2], and [Wei00].

3.1.1. *Algorithms and computability over the reals.*

Definition 3.1. Given $k \in \mathbb{N}$, we say that a function $f\colon \mathbb{N}^k \to \mathbb{Z}$ is *computable* if there exists an algorithm $\mathcal{A}$ such that, upon input a sequence of k positive integers $\{x_i\}_{i=1}^k$, it outputs the value of $f(x_1, x_2, \dots, x_k)$.

For a countable set S, by an *effective enumeration* of S we mean an enumeration $S = \{x_i\}_{i\in\mathbb{N}}$ satisfying that there exists an algorithm $\mathcal{A}$ which, upon input $i \in \mathbb{N}$, outputs x_i.

Definition 3.2. A real number x is called

(i) *computable* if there exist two computable functions $f\colon \mathbb{N} \to \mathbb{Z}$ and $g\colon \mathbb{N} \to \mathbb{N}$ such that for each $n \in \mathbb{N}$, $\bigl|\frac{f(n)}{g(n)} - x\bigr| < 2^{-n}$;

(ii) *lower semi-computable* if there exist two computable functions $f\colon \mathbb{N} \to \mathbb{Z}$ and $g\colon \mathbb{N} \to \mathbb{N}$ such that $\{f(n)/g(n)\}_{n\in\mathbb{N}}$ is a strictly increasing sequence and converges to x as $n \to +\infty$; and

(iii) *upper semi-computable* if there exist two computable functions $f\colon \mathbb{N} \to \mathbb{Z}$ and $g\colon \mathbb{N} \to \mathbb{N}$ such that $\{f(n)/g(n)\}_{n\in\mathbb{N}}$ is a strictly decreasing sequence and converges to x as $n \to +\infty$.

Moreover, we say that a sequence $\{x_i\}_{i\in\mathbb{N}}$ of real numbers is *a sequence of uniformly lower (resp. upper) semi-computable real numbers* if there exist two computable functions $f\colon \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ and $g\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for each $i \in \mathbb{N}$, $\{f(i, n)/g(i, n)\}_{n\in\mathbb{N}}$ is strictly increasing (resp. decreasing) in n and converges to x_i as $n \to +\infty$.

3.1.2. *Computable metric spaces.* Definition 3.2 equips $\mathbb{R}$ with a computability structure. Indeed, such a structure can be defined similarly to any separable metric space.

Definition 3.3. A *computable metric space* is a triple $(X,\,\rho,\,\mathcal{S})$, where

(i) (X,ρ) is a separable metric space;
(ii) $\mathcal{S}=\{s_n : n\in\mathbb{N}\}$ is a countable dense subset of X; and
(iii) there exists an algorithm which, on input $i,\,j,\,m\in\mathbb{N}$, outputs $y_{i,j,m}\in\mathbb{Q}$ satisfying $|y_{i,j,m}-\rho(s_i,s_j)|<2^{-m}$.

The points in $\mathcal{S}$ are said to be *ideal*. Due to the existence of computable bijection between $\mathbb{N}^3$ and $\mathbb{N}$, there exists an effective enumeration $\{B_l\}_{l\in\mathbb{N}}$ of the set $\{B(s_i,j/k) : i,\,j,\,k\in\mathbb{N}\}$ of balls with rational radii centered at points in $\mathcal{S}$. Specifically, there exists an algorithm that, given an input $l\in\mathbb{N}$, outputs the lower index of the ideal center and the rational radius of the ball B_l. These balls are called the *ideal balls* in $(X,\,\rho,\,\mathcal{S})$. We fix such an effective enumeration of ideal balls and call it the effective enumeration of ideal balls in $(X,\,\rho,\,\mathcal{S})$.

By the expressions of the chordal metric σ (see Section 2), we establish the following result as an example.

Example 3.4. Let $\mathbb{Q}(\widehat{\mathbb{C}})$ denote the set $\{a+b\mathbf{i} : a,\,b\in\mathbb{Q}\}\cup\{\infty\}$. Then $\big(\widehat{\mathbb{C}},\,\sigma,\,\mathbb{Q}(\widehat{\mathbb{C}})\big)$ is a computable metric space, where σ is the chordal metric on $\widehat{\mathbb{C}}$.

Definition 3.5. Let $(X,\,\rho,\,\mathcal{S})$ be a computable metric space with $\mathcal{S}=\{s_i\}_{i\in\mathbb{N}}$. We say that a point $x\in X$ is *computable* if there exists a computable function $f\colon\mathbb{N}\to\mathbb{N}$ such that $\rho\big(s_{f(n)},x\big)<2^{-n}$ for each $n\in\mathbb{N}$. Moreover, a sequence of points $\{x_i\}_{i\in\mathbb{N}}$ is said to be a *sequence of uniformly computable points* if there exists a computable function $f\colon\mathbb{N}^2\to\mathbb{N}$ such that $\rho\big(s_{f(n,m)},x_m\big)<2^{-n}$ for all $n,\,m\in\mathbb{N}$.

As a remark, a finite sequence of computable points is always a sequence of uniformly computable points. Similarly, for the other definitions of computable objects detailed below, we will say that a finite sequence of computable objects is also a sequence of uniformly computable objects.

Definition 3.6. In a computable metric space $(X,\,\rho,\,\mathcal{S})$, an open set $U\subseteq X$ is called *lower semi-computable* if there is a computable function $f\colon\mathbb{N}\to\mathbb{N}$ such that $U=\bigcup_{n\in\mathbb{N}}B_{f(n)}$. Moreover, a family $\{U_i\}_{i\in\mathbb{N}}$ of lower-computable open sets is called a *sequence of uniformly lower semi-computable open sets* if there is a computable function $f\colon\mathbb{N}^2\to\mathbb{N}$ such that $U_i=\bigcup_{n\in\mathbb{N}}B_{f(i,n)}$ for each $i\in\mathbb{N}$.

Remark 3.7. Assume that $r\in\mathbb{R}^+$ is a lower semi-computable real number. Then by Definition 3.2, there exists a sequence $\{r_i\}_{i\in\mathbb{N}}$ of rational numbers that is strictly increasing in i and converges to r. Assume that $x\in X$ is a computable point. Then there exists a computable function $f\colon\mathbb{N}\to\mathbb{N}$ such that $\rho\big(s_{f(n)},x\big)<2^{-n}$ for each $n\in\mathbb{N}$. Hence, by Definition 3.6, $B_\rho(x,r)=\bigcup_{n,\,i\in\mathbb{N}}B_\rho(s_{f(n)},r_i-2^{-n})$ is a lower semi-computable open set.

By Definition 3.6 and the existence of computable bijections between $\mathbb{N}^2$ and $\mathbb{N}$, we obtain the following result.

Proposition 3.8. *Assume that $\{U_i\}_{i\in\mathbb{N}}$ is a sequence of uniformly lower semi-computable open sets in a computable metric space $(X,\,\rho,\,\mathcal{S})$, then $\bigcup_{i\in\mathbb{N}}U_i$ is a lower semi-computable open set.*

Before defining computable functions between computable metric spaces, we first recall the definition of oracles for a point in a computable metric space.

Definition 3.9. Let $(X,\,\rho,\,\mathcal{S})$ be a computable metric space with $\mathcal{S}=\{s_i\}_{i\in\mathbb{N}}$ and x be a point in X. We say that a function $\varphi\colon\mathbb{N}\to\mathbb{N}$ is an *oracle* for $x\in X$ if $\rho\big(s_{\varphi(n)},x\big)<2^{-n}$ for each $n\in\mathbb{N}$.

Definition 3.10. Let $(X,\,\rho,\,\mathcal{S})$ and $(X',\,\rho',\,\mathcal{S}')$ be computable metric spaces with $\mathcal{S}=\{s_i\}_{i\in\mathbb{N}}$ and with $\mathcal{S}'=\{s'_i\}_{i\in\mathbb{N}}$, and let C be a subset of X. A function $f\colon X\to X'$ is said to be *computable on C* if there exists an algorithm such that for each $x\in X$ and each $n\in\mathbb{N}$, on input $n\in\mathbb{N}$ and an oracle φ for $x\in C$, outputs $m\in\mathbb{N}$ satisfying that $\rho'(s'_m,f(x))<2^{-n}$. Moreover, a sequence $\{f_i\}_{i\in\mathbb{N}}$ of functions $f_i\colon X\to X'$

is called a *sequence of uniformly computable functions on* C if there exists an algorithm such that for each $x \in X$, each $i \in \mathbb{N}$, and each $n \in \mathbb{N}$, on input $i, n \in \mathbb{N}$, and an oracle φ for $x \in C$, outputs $m \in \mathbb{N}$ satisfying that $\rho'(s'_m, f_i(x)) < 2^{-n}$. As a convention, we say that f is computable if f is computable on X.

For example, in [Wei00], Examples 4.3.3 and 4.3.13.5 show that the exponential function $\exp\colon \mathbb{R} \to \mathbb{R}$ and the logarithmic function $\log\colon \mathbb{R}^+ \to \mathbb{R}$ are both computable functions. The following proposition is a classical result that describes a topological property of computable functions. For our purpose, we state it in the following form and include the proof for the sake of completeness.

Proposition 3.11. *Let $(X, \rho, \mathcal{S})$ and $(X', \rho', \mathcal{S}')$ be computable metric spaces, C be a subset of X, and $\{B'_i\}_{i\in\mathbb{N}}$ be the effective enumeration of ideal balls in $(X', \rho', \mathcal{S}')$. Assume that $\{f_i\}_{i\in\mathbb{N}}$ is a sequence of functions $f_i\colon X \to X'$. Then $\{f_i\}_{i\in\mathbb{N}}$ is a sequence of uniformly computable functions on C if and only if there exists a sequence $\{U_{i,j} : i, j \in \mathbb{N}\}$ of uniformly lower semi-computable open sets in the computable metric space $(X, \rho, \mathcal{S})$ satisfying that $f_j^{-1}(B'_i) \cap C = U_{i,j} \cap C$ for each pair of $i, j \in \mathbb{N}$.*

Proof. Let $\mathcal{S} = \{s_i\}_{i\in\mathbb{N}}$ and $\mathcal{S}' = \{s'_i\}_{i\in\mathbb{N}}$. Now we assume that $\{f_i\}_{i\in\mathbb{N}}$ is a sequence of uniformly computable functions on C and show that there exists a sequence $\{U_{i,j} : i, j \in \mathbb{N}\}$ of uniformly lower semi-computable open sets in the computable metric space $(X, \rho, \mathcal{S})$ satisfying that $f_j^{-1}(B'_i) \cap C = U_{i,j} \cap C$ for each pair of $i, j \in \mathbb{N}$. For each $q \in \mathbb{N}$, we say that a sequence $\{p_i\}_{i=1}^{q}$ of integers is admissible in the computable metric space $(X, \rho, \mathcal{S})$ if $\rho\bigl(s_{p_{i+1}}, s_{p_i}\bigr) < 2^{-i-1}$ for each $i \in \mathbb{N}$. Note that $(X, \rho, \mathcal{S})$ is a computable metric space. By enumerating all the sequences of finitely many integers, it is not difficult to use Definition 3.3 (iii) to check the admissibility of sequences in order to obtain an effective enumeration $\{P_i\}_{i\in\mathbb{N}}$ of admissible sequences in $(X, \rho, \mathcal{S})$. Moreover, for each admissible sequence $P = \{p_i\}_{i=1}^{q}$, the function $\varphi_P\colon \mathbb{N} \to \mathbb{N}$ given by

$$\varphi_P(i) \coloneqq \begin{cases} p_i & \text{if } 1 \leqslant i \leqslant q; \\ p_q & \text{if } i \geqslant q+1. \end{cases}$$

is an oracle for the point $s_{p_q} \in X$.

Since $\{f_i\}_{i\in\mathbb{N}}$ is a sequence of uniformly computable functions on C, there exists an algorithm $M(\cdot,\cdot,\cdot)$ that satisfies that for each $x \in C$, each $n \in \mathbb{N}$, each $i \in \mathbb{N}$, and each oracle φ for x, $M(i,n,\varphi)$ outputs $m \in \mathbb{N}$ satisfying that $\rho'(s'_m, f_i(x)) < 2^{-n}$. We enumerate $\mathbb{N} \times \mathbb{N}$ by $\{(a_u, n_u)\}_{u\in\mathbb{N}}$ effectively. Now we design an algorithm $M'(\cdot,\cdot)$ which, for each pair of $i, j \in \mathbb{N}$, on input $i, j \in \mathbb{N}$, outputs a sequence $\{c_{i,j,k}\}_{k\in\mathbb{N}}$ of integers and a sequence $\{r_{i,j,k}\}_{k\in\mathbb{N}}$ of rational numbers satisfying that $f_j^{-1}(B'_i) \cap C = \bigcup_{k\in\mathbb{N}} \bigl(B_\rho\bigl(s_{c_{i,j,k}}, r_{i,j,k}\bigr) \cap C\bigr)$ for each $i, j \in \mathbb{N}$ as follows.

Begin

- (i) Read in the integers i and j.
- (ii) Set u and k both to be 1, and $\mathrm{flag}_i = 0$ for each $i \in \mathbb{N}$.
- (iii) **While** $u \geqslant 1$ **do**
  - (1) Run the algorithm $M\bigl(j, n_u, \varphi_{P_{a_u}}\bigr)$.
  - (2) Set v to be 1.
  - (3) **While** $1 \leqslant v \leqslant u$ **do**
    - (A) If
      - (a) flag_v equals to 0,
      - (b) the algorithm $M\bigl(j, n_v, \varphi_{P_{a_v}}\bigr)$ halts and outputs $m_v \in \mathbb{N}$ satisfying that
        $$B_{\rho'}\bigl(s'_{m_v}, 2^{-n_v}\bigr) \subseteq B'_i$$
        (the algorithm $M\bigl(j, n_v, \varphi_{P_{a_v}}\bigr)$ terminates after finitely many steps, and hence the oracle $\varphi_{P_{a_v}}$ is only quired up to some finite precision 2^{-w_v}),

      then
      - (a′) the algorithm $M'(i,j)$ outputs $c_{i,j,k} \coloneqq \varphi_{P_{a_v}}(w_v)$ and $r_{i,j,k} \coloneqq 2^{-w_v}$,

(b′) set flag_v to be 1 and k to be $k+1$.

(B) Set v to be $v+1$.

(4) Set u to be $u+1$.

End

Now we fix an pair of i, $j \in \mathbb{N}$, and verify that $f_j^{-1}(B'_i) \cap C = \bigcup_{k\in\mathbb{N}} \bigl(B_\rho\bigl(s_{c_{i,j,k}}, r_{i,j,k}\bigr) \cap C\bigr)$.

First, we fix an integer k and show that $B_\rho\bigl(s_{c_{i,j,k}}, r_{i,j,k}\bigr) \cap C \subseteq f_j^{-1}(B'_i)$. By Step (iii) (3) (A) of the algorithm $M'(i,j)$, we obtain that $c_{i,j,k} = \varphi_{P_{a_v}}(w_v)$ and $r_{i,j,k} = 2^{-w_v}$ for some $v \in \mathbb{N}$ with $B_{\rho'}\bigl(s'_{m_v}, 2^{-n_v}\bigr) \subseteq B'_i$. Here m_v is the output of the algorithm $M\bigl(j, n_v, \varphi_{P_{a_v}}\bigr)$. Note that $\mathcal{S} = \{s_i\}_{i\in\mathbb{N}}$ is dense in X. It is not hard to see that, for each $x \in B_\rho\bigl(s_{c_{i,j,k}}, r_{i,j,k}\bigr) \cap C$, there is a valid oracle $\widetilde{\varphi}_x$ that agrees with $\varphi_{P_{a_v}}$ up to precision 2^{-w_v}. Thus for each $x \in B_\rho\bigl(s_{c_{i,j,k}}, r_{i,j,k}\bigr) \cap C$, $M(j,\, n_v,\, \widetilde{\varphi}_x)$ outputs the same answer m_v and hence, we must have $f_j(x) \in B_{\rho'}\bigl(s'_{m_v}, 2^{-n_v}\bigr) \subseteq B'_i$. Then we have $f_j\bigl(B_\rho\bigl(s_{c_{i,j,k}}, r_{i,j,k}\bigr) \cap C\bigr) \subseteq B'_i$. Therefore, we obtain that $\bigcup_{k\in\mathbb{N}} \bigl(B_\rho\bigl(s_{c_{i,j,k}}, r_{i,j,k}\bigr) \cap C\bigr) \subseteq f_j^{-1}(B'_i) \cap C$.

Next, we establish that $\bigcup_{k\in\mathbb{N}} \bigl(B_\rho\bigl(s_{c_{i,j,k}}, r_{i,j,k}\bigr) \cap C\bigr) \supseteq f_j^{-1}(B'_i) \cap C$. Now we fix an point $x \in f_j^{-1}(B'_i) \cap C$, and show that $x \in B_\rho\bigl(s_{c_{i,j,k}}, r_{i,j,k}\bigr)$ for some $k \in \mathbb{N}$. Indeed, since $f_j(x) \in B'_i$, there exists $n(x) \in \mathbb{N}$ satisfying that $B_{\rho'}\bigl(f_j(x), 2^{-n(x)+1}\bigr) \subseteq B'_i$. Note that $\mathcal{S}$ is dense in X. It is not hard to see that, for $x \in C$, there is a valid oracle $\overline{\varphi}_x$ that satisfies that $\{\overline{\varphi}_x(v)\}_{v=1}^{q}$ is an admissible sequence for each $q \in \mathbb{N}$. Note that $x \in C$. Then the algorithm $M(j, n(x), \overline{\varphi}_x)$ will halt eventually. Assume that the output of the algorithm $M(j, n(x), \overline{\varphi}_x)$ is $m(x)$. Then $\rho'\bigl(s'_{m(x)}, f_j(x)\bigr) < 2^{-n(x)}$. Hence, $B_{\rho'}\bigl(s'_{m(x)}, 2^{-n(x)}\bigr) \subseteq B_{\rho'}\bigl(f_j(x), 2^{-n(x)+1}\bigr) \subseteq B'_i$. Assume that the oracle $\overline{\varphi}_x$ is only quired up to the precision $2^{-w(x)}$ by the algorithm $M(j, n(x), \overline{\varphi}_x)$. Denote the sequence $\{\overline{\varphi}_x(v)\}_{v=1}^{w(x)}$ by $Q(x)$. Then $Q(x)$ is an admissible sequence and the oracle $\varphi_{Q(x)}$ agrees with $\overline{\varphi}_x$ up to precision $2^{-w(x)}$. Thus $M\bigl(j,\, n(x),\, \varphi_{Q(x)}\bigr)$ outputs the same answer $m(x) \in \mathbb{N}$ as $M(j,\, n(x),\, \overline{\varphi}_x)$. Since $Q(x)$ is an admissible sequence, we will run the algorithm $M\bigl(j,\, n(x),\, \varphi_{Q(x)}\bigr)$ in Step (iii) (1) of the algorithm $M'(i,j)$. Since $B_{\rho'}\bigl(s'_{m(x)}, 2^{-n(x)}\bigr) \subseteq B'_i$, in Step (iii) (3) (A) of the algorithm $M'(i,j)$, $M'(i,j)$ will output $c_{i,j,k} = \varphi_{Q(x)}(w(x)) = \overline{\varphi}_x(w(x))$ and $r_{i,j,k} = 2^{-w(x)}$ for some $k \in \mathbb{N}$. Note that $\overline{\varphi}_x$ is an oracle for x. Then we have $x \in B_\rho\bigl(s_{\overline{\varphi}_x(w(x))}, 2^{-w(x)}\bigr) = B_\rho\bigl(s_{c_{i,j,k}}, r_{i,j,k}\bigr)$.

Hence, $f_j^{-1}(B'_i) \cap C = \bigl(\bigcup_{k\in\mathbb{N}} B_\rho\bigl(s_{c_{i,j,k}}, r_{i,j,k}\bigr)\bigr) \cap C$ for each pair of i, $j \in \mathbb{N}$. Note that by the existence of the algorithm $M'(\cdot,\cdot)$, we have $\bigl\{B_\rho\bigl(s_{c_{i,j,k}}, r_{i,j,k}\bigr) : i,\, j,\, k \in \mathbb{N}\bigr\}$ is a sequence of uniformly lower-computable open sets in the computable metric space $(X,\, \rho,\, \mathcal{S})$. From Definition 3.6, by constructing a computable bijection between $\mathbb{N}^3$ and $\mathbb{N}^2$, it is not hard to derive that $\bigl\{\bigcup_{k\in\mathbb{N}} B_\rho\bigl(s_{c_{i,j,k}}, r_{i,j,k}\bigr) : i,\, j \in \mathbb{N}\bigr\}$ is a sequence of uniformly lower-computable open sets in the computable metric space $(X,\, \rho,\, \mathcal{S})$.

Finally, we assume that there exists a sequence $\{U_{i,j} : i,\, j \in \mathbb{N}\}$ of uniformly lower semi-computable open sets in the computable metric space $(X,\, \rho,\, \mathcal{S})$ satisfying that $f_j^{-1}(B'_i) \cap C = U_{i,j} \cap C$ for each pair of i, $j \in \mathbb{N}$ and establish that $\{f_i\}_{i\in\mathbb{N}}$ is a sequence of uniformly computable functions on C. Now we fix an oracle φ_x of a point $x \in C$ and a pair of i, $n \in \mathbb{N}$. By Definition 3.10, it suffices to compute an integer m satisfying that $\rho'(s'_m, f_i(x)) < 2^{-n}$, i.e., $x \in f_i^{-1}(B_{\rho'}(s'_m, 2^{-n}))$.

Indeed, by hypotheses, we can compute a sequence $\{U_m\}_{m\in\mathbb{N}}$ of lower semi-computable open sets satisfying that $f_i^{-1}(B_{\rho'}(s'_m, 2^{-n})) \cap C = U_m \cap C$ for each $m \in \mathbb{N}$. Note that $x \in C$. Then $x \in f_i^{-1}(B_{\rho'}(s'_m, 2^{-n}))$ if and only if $x \in U_m$, i.e., $B_\rho\bigl(s_{\varphi_x(t)}, 2^{-t}\bigr) \subseteq U_m$ for some $t \in \mathbb{N}$. By the uniform lower semi-computable openness of the sequence $\{U_m\}_{m\in\mathbb{N}}$, it is not hard to construct an algorithm which, on input $m \in \mathbb{N}$, halts if and only if $x \in U_m$. Note that $\mathcal{S} = \{s_m\}_{m\in\mathbb{N}}$ is dense in X. Then there exists an integer m satisfying that $x \in U_m$. Therefore, we can find an integer $m \in \mathbb{N}$ such that $x \in U_m$ for each $x \in X$. Therefore, we establish that $\{f_i\}_{i\in\mathbb{N}}$ is a sequence of uniformly computable functions on C. □

Definition 3.12. Let $(X,\, \rho,\, \mathcal{S})$ be a computable metric space and let C be a subset of X. A function $f\colon X \to \mathbb{R}$ is said to be *upper semi-computable* (resp. *lower semi-computable*) on C if there exists a sequence $f_i\colon X \to \mathbb{R}$, $i \in \mathbb{N}$, of uniformly computable functions on C such that for each $x \in C$, $\{f_i(x)\}_{i\in\mathbb{N}}$ is non-increasing (resp. non-decreasing) in i and $f_i(x) \to f(x)$ as $i \to +\infty$.

The following proposition is an immediate corollary of Proposition 3.11.

Proposition 3.13. *Fix an effective enumeration $\{q_n\}_{n\in\mathbb{N}}$ of $\mathbb{Q}$. Let $(X,\,\rho,\,\mathcal{S})$ be a computable metric space, C a subset of X. If $f\colon X\to\mathbb{R}$ is upper semi-computable (resp. lower semi-computable) on C, then there is a sequence $\{U_i\}_{i\in\mathbb{N}}$ of uniformly lower semi-computable open sets in the computable metric space $(X,\,\rho,\,\mathcal{S})$ satisfying that for each $i\in\mathbb{N}$,*

$$f^{-1}((-\infty,q_i))\cap C=U_i\cap C\quad (\textit{resp. } f^{-1}((q_i,+\infty))\cap C=U_i\cap C).$$

Proof. By Definition 3.12, it suffices to verify the case where f is an upper semi-computable function on C. It follows from the upper semi-computability of f on C that there exists a sequence $f_i\colon X\to\mathbb{R}$, $i\in\mathbb{N}$, of uniformly computable functions on C such that for each $x\in C$, $\{f_i(x)\}_{i\in\mathbb{N}}$ is non-increasing in i and $f_i(x)\to f(x)$ as $i\to+\infty$. Hence, we obtain that $f^{-1}((-\infty,q_i))\cap C=\bigcup_{j\in\mathbb{N}}\bigl(f_j^{-1}((-\infty,q_i))\cap C\bigr)$ for each $i\in\mathbb{N}$. By Proposition 3.11, there exists a sequence $\{U_{i,j}:i,\,j\in\mathbb{N}\}$ of uniformly lower semi-computable open sets in the computable metric space $(X,\,\rho,\,\mathcal{S})$ satisfying that $f_j^{-1}((-\infty,q_i))\cap C=U_{i,j}\cap C$ for each pair of $i,\,j\in\mathbb{N}$. Therefore, we obtain that $f^{-1}((-\infty,q_i))\cap C=\bigcup_{j\in\mathbb{N}}(U_{i,j}\cap C)=\bigl(\bigcup_{j\in\mathbb{N}}U_{i,j}\bigr)\cap C$. By Proposition 3.8, $\bigcup_{j\in\mathbb{N}}U_{i,j}$ is a lower semi-computable open set for each $i\in\mathbb{N}$. □

Moreover, it follows directly from Definition 3.10 that computable real-valued functions are closed under a finite number of operations from the following list: addition, multiplication, division, scalar multiplication, maximum and minimum (see for example, [Wei00, Corollary 4.3.4]).

Finally, we recall the definitions of recursively compact sets and recursively precompact metric spaces as introduced in [GHR11, Section 2].

Definition 3.14. Let $(X,\,\rho,\,\mathcal{S})$ be a computable metric space with $\mathcal{S}=\{s_i\}_{i\in\mathbb{N}}$. Then a subset $K\subseteq X$ is said to be *recursively compact* if it is compact and there is an algorithm that, on input a sequence $\{i_j\}_{j=1}^p$ of integers and a sequence $\{q_j\}_{j=1}^p$ of positive rational numbers, halts if and only if $K\subseteq\bigcup_{j=1}^p B\bigl(s_{i_j},q_j\bigr)$.

Definition 3.15. Let $(X,\,\rho,\,\mathcal{S})$ be a computable metric space with $\mathcal{S}=\{s_i\}_{i\in\mathbb{N}}$. Then $(X,\,\rho,\,\mathcal{S})$ is said to be recursively precompact if there exists an algorithm which, on input $n\in\mathbb{N}$, outputs a finite set $\{i_1,\,\dots,\,i_p\}\subseteq\mathbb{N}$ such that $X=\bigcup_{j=1}^p B_\sigma(s_{i_j},2^{-n})$.

Here we recall [GHR11, Proposition 4].

Proposition 3.16 (Galatolo, Hoyrup, & Rojas [GHR11])**.** *Let $(X,\,\rho,\,\mathcal{S})$ be a computable metric space. Then X is recursively compact if and only if (X,ρ) is complete and $(X,\,\rho,\,\mathcal{S})$ is recursively precompact.*

Example 3.17. Consider the computable metric space $(\widehat{\mathbb{C}},\,\sigma,\,\mathbb{Q}(\widehat{\mathbb{C}}))$ with $\mathbb{Q}(\widehat{\mathbb{C}})=\{s_i\}_{i\in\mathbb{N}}$. By Definition 3.15, it is not hard to see that $(\widehat{\mathbb{C}},\,\sigma,\,\mathbb{Q}(\widehat{\mathbb{C}}))$ is recursively precompact. Since $(\widehat{\mathbb{C}},\sigma)$ is complete, by Proposition 3.16, $\widehat{\mathbb{C}}$ is a recursively compact set in the computable metric space $(\widehat{\mathbb{C}},\,\sigma,\,\mathbb{Q}(\widehat{\mathbb{C}}))$.

The following are some fundamental properties of recursively compact sets as discussed in [GHR11, Proposition 1].

Proposition 3.18 (Galatolo, Hoyrup, & Rojas [GHR11])**.** *Let $(X,\,\rho,\,\mathcal{S})$ be a computable metric space. Assume that $K\subseteq X$ is a recursively compact set. Then the following statements hold:*

(i) *A point $x\in X$ is computable if and only if the singleton $\{x\}$ is a recursively compact set.*
(ii) *$X\smallsetminus K$ is a lower semi-computable open set.*
(iii) *If $U\subseteq X$ is a lower semi-computable open set, then $K\smallsetminus U$ is recursively compact.*
(iv) *If $f\colon X\to\mathbb{R}$ is lower semi-computable, then $\inf_{x\in K}f(x)$ is lower semi-computable.*
(v) *If $f\colon X\to\mathbb{R}$ is upper semi-computable, then $\sup_{x\in K}f(x)$ is upper semi-computable.*
(vi) *If $K'\subseteq X$ is a recursively compact set, then so is $K'\cap K$.*

3.1.3. *Computability of probability measures.* In this subsection, we give a computable structure on $\mathcal{P}(X)$ and introduce some related results.

Assume that X is a recursively compact set and $(X, \rho, \mathcal{S})$ is a computable metric space. Let W_ρ denote the Wasserstein–Kantorovich metric of $\mathcal{P}(X)$ (recalled in (2.1)), and $\mathcal{Q}_\mathcal{S} \subseteq \mathcal{P}(X)$ denote the set of the Borel probability measures supported on finitely many points in $\mathcal{S}$ assigning rational values on them. Then it follows from [HR09, Proposition 4.1.3] that $\big(\mathcal{P}(X), W_\rho, \mathcal{Q}_\mathcal{S}\big)$ is a computable metric space. We remark that the definition of computable metric spaces adopted in [HR09] additionally requires the completeness of the space X, compared to Definition 3.3.

Definition 3.19. Let $(X, \rho, \mathcal{S})$ be a computable metric space, X a recursively compact set, and $\mu \in \mathcal{P}(X)$ be a Borel probability measure. Then we say that μ is a *computable measure* if μ is a computable point of $\big(\mathcal{P}(X), W_\rho, \mathcal{Q}_\mathcal{S}\big)$.

By [GHR11, Lemma 2.12] and Proposition 3.16, due to the completeness of $\mathcal{P}(X)$ with respect to the metric W_ρ, one can conclude the following result.

Proposition 3.20. *Let $(X, \rho, \mathcal{S})$ be a computable metric space, and X a recursively compact set. Then $\mathcal{P}(X)$ is a recursively compact set in $\big(\mathcal{P}(X), W_\rho, \mathcal{Q}_\mathcal{S}\big)$. Moreover, for each recursively compact subset $K \subseteq X$, $\mathcal{P}(X, K)$ is a recursively compact set.*

Assume that X is recursively compact. Then by [BRY14, Proposition 2.13], upper bounds of computable functions on X can be computed. Therefore, the following result follows from [HR09, Corollary 4.3.2].

Corollary 3.21. *Let $(X, \rho, \mathcal{S})$ be a computable metric space, and X a recursively compact set. Assume that $f_i \colon X \to \mathbb{R}$, $i \in \mathbb{N}$, is a sequence of uniformly computable functions on X. For each $i \in \mathbb{N}$, we denote by $\mathcal{I}_i \colon \mathcal{P}(X) \to \mathbb{R}$ the integral operator defined by $\mathcal{I}_i(\mu) \coloneqq \int f_i \,\mathrm{d}\mu$ for each $\mu \in \mathcal{P}(X)$. Then $\{\mathcal{I}_i\}_{i\in\mathbb{N}}$ is a sequence of uniformly computable functions on $\mathcal{P}(X)$.*

Finally, we introduce a family of computable functions.

Definition 3.22. Let (X, ρ) be a metric space. Consider arbitrary constants r, $\epsilon > 0$, and a point $u \in X$. Then the function $g_{u,r,\epsilon}$ given by

$$g_{u,r,\epsilon}(x) \coloneqq \left|1 - \frac{|\rho(x,u) - r|^+}{\epsilon}\right|^+ \quad \text{for each } x \in X, \tag{3.1}$$

is called a *hat function*.

The hat function $g_{u,r,\epsilon}(x)$ is an ϵ^{-1}-Lipschitz function that equals to 1 within the ball $B(u, r)$, 0 outside the ball $B(u, r + \epsilon)$, and lies strictly between 0 and 1 in the annulus $B(u, r + \epsilon) \smallsetminus \overline{B(u, r)}$.

Using hat functions, we can introduce the following result.

Proposition 3.23. *Let $(X, \rho, \mathcal{S})$ be a computable metric space. Assume that $\{U_j\}_{j\in\mathbb{N}}$ is a sequence of uniformly lower semi-computable open sets. Then there exists a sequence $h_{j,k} \colon X \to \mathbb{R}$, j, $k \in \mathbb{N}$, of uniformly computable functions satisfying that for each $j \in \mathbb{N}$,*

(i) *$\{h_{j,k}(x)\}_{k\in\mathbb{N}}$ is non-decreasing in k and $h_{j,k}(x) \to \mathbb{1}_{U_j}(x)$ as $k \to +\infty$ for each $x \in X$,*
(ii) *$h_{j,k}(x) = 0$ for each $x \notin U_j$ and each $k \in \mathbb{N}$.*

Proof. Let $\{q_i\}_{i\in\mathbb{N}}$ be an effective enumeration of $\mathbb{Q}$ and $\mathcal{S} = \{s_i\}_{i\in\mathbb{N}}$. By Definition 3.6, there exists a pair of computable functions $f \colon \mathbb{N}^2 \to \mathbb{N}$ and $l \colon \mathbb{N}^2 \to \mathbb{N}$ such that $U_j = \bigcup_{n\in\mathbb{N}} B\big(s_{f(j,n)}, q_{l(j,n)}\big)$ for each $j \in \mathbb{N}$. For each $j \in \mathbb{N}$ and $k \in \mathbb{N}$, we now define the function $h_{j,k} \colon X \to \mathbb{R}$ by using (3.1) as follows:

$$h_{j,k}(x) = \max\Big\{ g_{s_{f(j,i)}, q_{l(j,i)} - \frac{1}{k}, \frac{1}{k}}(x) : i \in \mathbb{N} \text{ and } 1 \leqslant i \leqslant k \Big\} \quad \text{for each } x \in X.$$

Combining with Definition 3.22, it is not hard to see that the sequence $h_{j,k} \colon X \to \mathbb{R}$, j, $k \in \mathbb{N}$, of uniformly computable functions satisfies the requirements in Proposition 3.23. $\square$

Definition 3.24. In a computable metric space $(X, \rho, \mathcal{S})$, let $\mathcal{F}_0(\mathcal{S})$ be the set of functions of the form $g_{u,r,1/n}(x)$, where $u \in \mathcal{S}$, $r \in \mathbb{Q}^+$, and $n \in \mathbb{N}$. Let $\mathfrak{E}(\mathcal{S})$ be the smallest set of functions containing $\mathcal{F}_0$ and the constant function 1, closed under maximum, minimum, and finite rational linear combinations. The elements in $\mathfrak{E}(\mathcal{S})$ are called *test functions*.

Remark 3.25. Note that there exists a computable bijection between N^* and N. Hence, from Definitions 3.22 and 3.24, it is not hard to construct an effective enumeration $\{\varphi_j\}_{j\in\mathbb{N}}$ of $\mathfrak{E}(\mathcal{S})$, i.e., $\{\varphi_j\}_{j\in\mathbb{N}}$ is a sequence of uniformly computable functions. We fix such an effective enumeration of $\mathfrak{E}(\mathcal{S})$ and call it the *effective enumeration* of $\mathfrak{E}(\mathcal{S})$ in $(X, \rho, \mathcal{S})$.

Proposition 3.26. *Let $(X, \rho, \mathcal{S})$ be a computable metric space, X a recursively compact set, and $\{\varphi_j\}_{j\in\mathbb{N}}$ be the effective enumeration of $\mathfrak{E}(\mathcal{S})$. Then $\mathfrak{E}(\mathcal{S})$ is dense in the space $C(X)$ of continuous functions on X. Moreover, for each pair of $\mu, \nu \in \mathcal{M}(X)$, we have $\mu(A) \geqslant \nu(A)$ for each $A \in \mathcal{B}(X)$ if and only if $\langle \mu, \varphi_j\rangle \geqslant \langle \nu, \varphi_j\rangle$ for each $j \in \mathbb{N}$.*

Proof. By Stone–Weierstrass theorem (see for example, [Fol13, Theorem 4.45]), it immediately follows from Definition 3.24 that $\mathfrak{E}(\mathcal{S})$ is dense in $C(X)$. Hence, by the Dominated Convergence Theorem, we conclude that $\langle \mu, \varphi_j\rangle \geqslant \langle \nu, \varphi_j\rangle$ for each $j \in \mathbb{N}$ if and only if $\langle \mu, \varphi\rangle \geqslant \langle \mu, \varphi\rangle$ for each $\varphi \in C(X)$. Note that $\mathcal{M}(X)$ is the dual space of $C(X)$. Then $\langle \mu, \varphi\rangle \geqslant \langle \mu, \varphi\rangle$ for each $\varphi \in C(X)$ if and only if $\mu(A) \geqslant \nu(A)$ for each $A \in \mathcal{B}(X)$. □

3.2. **Thermodynamic formalism.** We first review some basic concepts from the ergodic theory and dynamical systems. We refer the reader to [Wal82, Chapter 9], or [KH95, Chapter 20] for more detailed studies of these concepts.

Let (X, d) be a compact metric space and $g\colon X \to X$ a continuous map. Given $n \in \mathbb{N}$,

$$d_g^n(x, y) := \max\bigl\{d\bigl(g^k(x), g^k(y)\bigr) : k \in \{0, 1, \dots, n-1\}\bigr\}, \qquad \text{for } x, y \in X,$$

defines a metric on X. A set $F \subseteq X$ is (n, ϵ)-*separated* (with respect to g), for some $n \in \mathbb{N}$ and $\epsilon > 0$, if for each pair of distinct points $x, y \in F$, we have $d_g^n(x, y) \geqslant \epsilon$. Given $\epsilon > 0$ and $n \in \mathbb{N}$, let $F_n(\epsilon)$ be a maximal (in the sense of inclusion) (n, ϵ)-separated set in X.

For each real-valued continuous function $\psi \in C(X)$, the following limits exist and are equal, and we denote these limits by $P(g, \psi)$ (see for example, [Wal82, Theorem 9.4 (viii)]):

$$P(g, \psi) := \lim_{\epsilon\to 0^+} \limsup_{n\to+\infty} \frac{1}{n} \log \sum_{x\in F_n(\epsilon)} \exp(S_n\psi(x)), \tag{3.2}$$

where $S_n\psi(x) := \sum_{j=0}^{n-1} \psi\bigl(g^j(x)\bigr)$ for all $n \in \mathbb{N}$ and $x \in X$. We call $P(g, \psi)$ the *topological pressure* of g with respect to the *potential* ψ. Note that $P(g, \psi)$ is independent of d as long as the topology on X defined by d remains the same (see for example, [Wal82, Section 9.1]). The quantity $h_{\mathrm{top}}(g) := P(g, 0)$ is called the *topological entropy* of g. The topological entropy is well-behaved under iterations. Indeed, if $n \in \mathbb{N}$, then $h_{\mathrm{top}}(g^n) = nh_{\mathrm{top}}(g)$ (see for example, [KH95, Proposition 3.1.7 (3)]).

A *measurable partition* ξ of X is a collection $\xi = \{A_i : i \in J\}$ consisting of countably many mutually disjoint sets in $\mathcal{B}$, where J is a countable (i.e., finite or countably infinite) index set. The measurable partition ξ is finite if the index set J is a finite set.

Let $\xi = \{A_j : j \in J\}$ and $\eta = \{B_k : k \in K\}$ be measurable partitions of X, where J and K are the corresponding index sets. We say ξ is a *refinement* of η if for each $A_j \in \xi$, there exists $B_k \in \eta$ such that $A_j \subseteq B_k$. The *common refinement* (or *join*) $\xi \vee \eta$ of ξ and η defined as

$$\xi \vee \eta := \{A_j \cap B_k : j \in J, k \in K\}$$

is also a measurable partition. Put $g^{-1}(\xi) := \bigl\{g^{-1}(A_j) : j \in J\bigr\}$, and for each $n \in \mathbb{N}$ define

$$\xi_g^n := \bigvee_{j=0}^{n-1} g^{-j}(\xi) = \xi \vee g^{-1}(\xi) \vee \cdots \vee g^{-(n-1)}(\xi).$$

Let $\xi = \{A_j : j \in J\}$ be a measurable partition of X and $\mu \in \mathcal{M}(X,g)$ be a g-invariant Borel probability measure on X. The *entropy* of ξ is $H_\mu(\xi) \coloneqq -\sum_{j\in J} \mu(A_j) \log(\mu(A_j)) \in [0,+\infty]$, where $0 \log 0$ is defined to be zero. One can show that (see for example, [Wal82, Chapter 4]) if $H_\mu(\xi) < +\infty$, then the following limit exists:

$$h_\mu(g,\xi) \coloneqq \lim_{n\to+\infty} \frac{1}{n} H_\mu(\xi_g^n) \in [0,+\infty). \tag{3.3}$$

The quantity $h_\mu(g,\xi)$ is called the *measure-theoretic entropy of g relative to ξ*. The *measure-theoretic entropy* of g for μ is defined as

$$h_\mu(g) \coloneqq \sup\{h_\mu(g,\xi) : \xi \text{ is a measurable partition of } X \text{ with } H_\mu(\xi) < +\infty\}. \tag{3.4}$$

For each real-valued continuous function $\psi \in C(X)$, the *measure-theoretic pressure* $P_\mu(g,\psi)$ of g for the measure $\mu \in \mathcal{M}(X,g)$ and the potential ψ is

$$P_\mu(g,\psi) \coloneqq h_\mu(g) + \int \psi \,\mathrm{d}\mu. \tag{3.5}$$

The topological pressure is related to the measure-theoretic pressure by the so-called *Variational Principle*. It states that (see for example, [Wal82, Theorem 9.10])

$$P(g,\psi) = \sup\{P_\mu(g,\psi) : \mu \in \mathcal{M}(X,g)\} \tag{3.6}$$

for each $\psi \in C(X)$. In particular, when ψ is the constant function 0,

$$h_{\mathrm{top}}(g) = \sup\{h_\mu(g) : \mu \in \mathcal{M}(X,g)\}. \tag{3.7}$$

A measure μ that attains the supremum in (3.6) is called an *equilibrium state* for the map g and the potential ψ. We denote by $\mathcal{E}(g,\psi)$ the set of equilibrium states for the map g and the potential ψ. A measure μ that attains the supremum in (3.7) is called a *measure of maximal entropy* of g.

4. Approach I

In this section, we establish the computability of equilibrium states for some dynamical systems whose upper semi-continuous measure-theoretic entropy functions. We begin by introducing some notations and results from functional analysis. Then we recall a relation between $C(X)^*_{\phi,P(T,\cdot)}$ and $\mathcal{E}(T,\phi)$ as described in (4.2). Finally, we establish the recursive compactness of $C(X)^*_{\phi,P(T,\cdot)}$ and finish the proof of Theorem 1.1.

We first recall some notations and results in functional analysis.

Definition 4.1. Assume that V is a real topological vector space, and $F\colon V \to \mathbb{R}$ is a convex continuous function. We say that a continuous linear functional $f\colon V \to \mathbb{R}$ *tangent* to F at $x \in V$ if

$$f(y) \leqslant F(x+y) - F(x)$$

for each $y \in V$. We denote the set of all such functionals by $V^*_{x,F}$.

Then we summarize two results from [Wal82, Theorem 9.7 (iv) & (v)] in the following lemma.

Lemma 4.2. *Assume that (X,ρ) is a compact metric space, $T\colon X \to X$ is a continuous map, and $P\colon C(X) \to \mathbb{R}$ is the function given by $P(\phi) \coloneqq P(T,\phi)$ for each $\phi \in C(X)$. Then P is convex and Lipschitz continuous with Lipschitz constant 1.*

Lemma 4.2 allows us to consider the set $C(X)^*_{\phi,P(T,\cdot)}$ for a continuous map on a compact metric space. Then we summarize [Wal92, Theorem 3 (i)] in the following lemma.

Lemma 4.3. *Assume that (X,ρ) is a compact metric space, $T\colon X \to X$ is a continuous map, and $\phi\colon X \to \mathbb{R}$ is a continuous function. Then for each functional $F \in C(X)^*$, we have $F \in C(X)^*_{\phi,P(T,\cdot)}$ if and only if F is represented by a T-invariant measure μ_F that is a weak*-limit of measures $\mu_n \in \mathcal{M}(X,T)$, $n \in \mathbb{N}$, such that $h_{\mu_n}(T) + \langle \mu_n, \phi\rangle \to P(T,\phi)$ as $n \to +\infty$.*

Here we say that $F \in C(X)^*$ *is represented by* $\mu_F \in \mathcal{P}(X)$ if and only if $F(f) = \int f \mathrm{d}\mu_F$ for each $f \in C(X)$. Lemma 4.3 allows us to see $C(X)^*_{\phi,P(T,\cdot\,)}$ as a subset of $\mathcal{M}(X,T)$.

For each $\mu \in \mathcal{M}(X,T)$, denote by $\overline{h}_\mu(T)$ the supremum of all accumulation points of sequences $\{h_{\mu_n}(T)\}_{n\in\mathbb{N}}$, where the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ ranges over all sequences of T-invariant Borel probability measures that tend to the measure μ in weak* topology.

Now we recall a relation between $C(X)^*_{\phi,P(T,\cdot\,)}$ and $\mathcal{E}(T,\phi)$ as follows.

Corollary 4.4. *Assume that* (X,ρ) *is a compact metric space, and* $T\colon X \to X$ *is a continuous map. Then*

$$C(X)^*_{\phi,P(T,\cdot\,)} = \big\{\mu \in \mathcal{M}(X,T) : \overline{h}_\mu(T) + \langle \mu, \phi\rangle = P(T,\phi)\big\} \text{ and} \tag{4.1}$$

$$\mathcal{E}(T,\phi) = C(X)^*_{\phi,P(T,\cdot\,)} \cap \big\{\mu \in \mathcal{M}(X,T) : \overline{h}_\mu(T) \leqslant h_\mu(T)\big\}. \tag{4.2}$$

In the above corollary, (4.1) follows immediately from Lemma 4.3, and (4.2) from [Wal92, Theorem 5]. Finally, we record [Wal82, Theorem 9.12].

Lemma 4.5. *Assume that* (X,ρ) *is a compact metric space, and* $T\colon X \to X$ *is a continuous map. Then*

$$\overline{h}_\mu(T) = \inf\{P(T,\theta) - \langle \mu, \theta\rangle : \theta \in C(X)\}. \tag{4.3}$$

With these preparations, we now establish Theorem 1.1.

Proof of Theorem 1.1. First, we show that the tangent space $C(X)^*_{\phi,P(T,\cdot\,)}$ is recursively compact. By (4.1) in Corollary 4.4 and (4.3) in Lemma 4.5, we have

$$C(X)^*_{\phi,P(T,\cdot\,)} = \bigg\{\mu \in \mathcal{M}(X,T) : \inf\bigg\{P(T,\theta) - \int \theta \,\mathrm{d}\mu : \theta \in C(X)\bigg\} = P(T,\phi) - \int \phi \,\mathrm{d}\mu\bigg\}.$$

Note that $\overline{D}$ contains a neighborhood of ϕ. By Lemma 4.2, $P(\cdot) := P(T,\cdot)$ is convex. Then for each $\mu \in \mathcal{M}(X,T)$, we obtain that

$$\inf\bigg\{P(T,\theta) - \int \theta \,\mathrm{d}\mu : \theta \in C(X)\bigg\} = P(T,\phi) - \int \phi \,\mathrm{d}\mu$$

is equivalent to

$$\inf\bigg\{P(T,\theta') - \int \theta' \,\mathrm{d}\mu : \theta' \in \overline{D}\bigg\} \geqslant P(T,\phi) - \int \phi \,\mathrm{d}\mu.$$

Moreover, since P is continuous and $D = \{\psi_i\}_{i\in\mathbb{N}}$, then for each $\mu \in \mathcal{M}(X,T)$,

$$\inf\bigg\{P(T,\theta') - \int \theta' \,\mathrm{d}\mu : \theta' \in \overline{D}\bigg\} = \inf\bigg\{P(T,\theta') - \int \theta' \,\mathrm{d}\mu : \theta' \in D\bigg\} = \inf_{i\in\mathbb{N}}\bigg\{P(T,\psi_i) - \int \psi_i \,\mathrm{d}\mu\bigg\}.$$

Hence, we obtain that

$$C(X)^*_{\phi,P(T,\cdot\,)} = \bigg\{\mu \in \mathcal{M}(X,T) : \inf_{i\in\mathbb{N}}\bigg\{P(T,\psi_i) - \int \psi_i \,\mathrm{d}\mu\bigg\} \geqslant P(T,\phi) - \int \phi \,\mathrm{d}\mu\bigg\}. \tag{4.4}$$

Define $f\colon \mathcal{P}(X) \to \mathbb{R}$ by

$$f(\nu) := \inf_{i\in\mathbb{N}}\bigg\{P(T,\psi_i) - \int \psi_i \,\mathrm{d}\nu\bigg\} + \int \phi \,\mathrm{d}\nu \quad \text{for each } \nu \in \mathcal{P}(X). \tag{4.5}$$

Claim. The function f is upper semi-computable on $\mathcal{P}(X)$.

Note that there exists an algorithm which, on input $i \in \mathbb{N}$, outputs a non-increasing sequence $\{p_{n,i}\}_{n\in\mathbb{N}}$ of real values tending to $P(T,\psi_i)$ (see property (i) in Theorem 1.1). Then by Corollary 3.21, it follows from the uniform computability of $\{\psi_i\}_{i\in\mathbb{N}}$ and the computability of ϕ that the sequence of integral functions $\mathcal{I}_i\colon \mathcal{P}(X) \to \mathbb{R}$, $i \in \mathbb{N}$, is a sequence of uniformly computable functions. Here $\mathcal{I}_i(\nu) := \int(\phi - \psi_i)\,\mathrm{d}\nu$ for each $i \in \mathbb{N}$ and each $\nu \in \mathcal{P}(X)$. Define the functions $F_n(\nu) := \inf_{1\leqslant i\leqslant n}\big\{p_{n,i} + \int(\phi - \psi_i)\,\mathrm{d}\nu\big\}$ for each $n \in \mathbb{N}$ and each $\nu \in \mathcal{P}(X)$. Then the sequence $\{F_n\}_{n\in\mathbb{N}}$ of functions is a sequence of uniformly computable functions. Note that $\{p_{n,i}\}_{n\in\mathbb{N}}$ is non-increasing in n. Then for each $\nu \in \mathcal{P}(X)$, $\{F_n(\nu)\}_{n\in\mathbb{N}}$ is non-increasing in n.

By Definition 3.12, to prove that f is an upper semi-computable function on $\mathcal{P}(X)$, it suffices to show that for each $\nu \in \mathcal{P}(X)$, $F_n(\nu) \to f(\nu)$ as $n \to +\infty$.

Now we fix a Borel probability measure $\nu \in \mathcal{P}(X)$. Since $\{F_n(\nu)\}_{n\in\mathbb{N}}$ is non-increasing and $F_n(\nu) \geqslant f(\nu)$ for each $n \in \mathbb{N}$, $\lim_{n\to+\infty} F_n(\nu)$ exists and is not smaller than $f(\nu)$. On the other hand, since $\lim_{n\to+\infty} p_{n,i} = P(T,\psi_i)$ for each $i \in \mathbb{N}$, it follows from (4.5) that, for each $\epsilon > 0$, we have $f(\nu) + \epsilon > P(T,\psi_i) + \int(\phi - \psi_i)\,\mathrm{d}\nu + \epsilon/2 > p_{j,i} + \int(\phi - \psi_i)\,\mathrm{d}\nu \geqslant F_{\max\{i,j\}}(\nu)$ for some $i \in \mathbb{N}$ and some $j \in \mathbb{N}$. Thus, $f(\nu) \geqslant \lim_{n\to+\infty} F_n(\nu)$. Then we obtain that $\lim_{n\to+\infty} F_n(\nu) = f(\nu)$ and hence, the claim follows.

Note that $P(T,\phi)$ is lower semi-computable. Then we can obtain a non-decreasing sequence of uniformly computable real values converging to $P(T,\phi)$, say $\{p_n\}_{n\in\mathbb{N}}$. Hence, it follows from (4.4) and (4.5) that

$$C(X)^*_{\phi,P(T,\cdot)} = \mathcal{M}(X,T) \cap f^{-1}([P(T,\phi),+\infty)) = \mathcal{M}(X,T) \smallsetminus \bigcup_{n\in\mathbb{N}} f^{-1}((-\infty,p_n)). \tag{4.6}$$

Then by the claim and the uniform computability of $\{p_n\}_{n\in\mathbb{N}}$, it follows from Proposition 3.13 that $\{f^{-1}((-\infty,p_n))\}_{n\in\mathbb{N}}$ is a sequence of uniformly lower semi-computable open sets. Hence, it follows from Proposition 3.8 that $\bigcup_{n\in\mathbb{N}} f^{-1}((-\infty,p_n))$ is a lower semi-computable open set. By [BHLZ24, Lemma 4.12], $\mathcal{M}(X,T)$ is recursively compact. Thus, by Proposition 3.18 (ii), $C(X)^*_{\phi,P(T,\cdot)}$ is a recursively compact set.

We now establish that the measure μ_ϕ is computable. Under the additional assumptions that the measure-theoretic entropy function $\nu \mapsto h_\nu(T)$ is upper semi-continuous on $\mathcal{M}(X,T)$ and $\mathcal{E}(T,\phi) = \{\mu_\phi\}$. By (4.2), it follows from $\overline{h}_\nu(T) = h_\nu(T)$ for each $\nu \in \mathcal{M}(X,T)$ that $C(X)^*_{\phi,P(T,\cdot)} = \mathcal{E}(T,\phi) = \{\mu_\phi\}$. Hence, by Proposition 3.18 (i), μ_ϕ is computable. $\square$

5. Approach II

In this section, we generalize the method in [BBRY11, BHLZ24] to establish Theorem 1.3.

5.1. **Jacobians and transfer operators.**
In this subsection, we first recall some important notions and results in ergodic theory. Then we define the transfer operators and their properties in our context. Using the transfer operators, we give an equivalent description of Jacobians on some subsets with respect to invariant measures in Theorem 5.5. We end this subsection with Corollary 5.6, which allows us to find equilibrium states by computing its Jacobians for some dynamical systems with some additional assumptions.

Definition 5.1 (Jacobian)**.** Let (X,ρ) be a compact metric space, $T\colon X \to X$ a continuous map, $\mu \in \mathcal{P}(X)$ a Borel probability measure on X, and $E \in \mathcal{B}(X)$ a Borel subset with full μ-measure. We say that a μ-measurable function $J\colon X \to [0,+\infty)$ is a *Jacobian (function)* on E for T with respect to μ if

$$\mu(T(A)) = \int_A J\,\mathrm{d}\mu \tag{5.1}$$

whenever $A \subseteq E$ is a μ-measurable subset, for which $T(A)$ is μ-measurable and T is injective on A. Moreover, we say that a μ-measurable non-negative function $J\colon X \to [0,\infty)$ is a *Jacobian (function)* for T with respect to μ if there exists a Borel subset E with full μ-measure that satisfies that J is a Jacobian on E for T with respect to μ.

Definition 5.2 (Singular point)**.** Let (X,ρ) be a compact metric space, and $T\colon X \to X$ be a continuous map that is finite-to-one, i.e., the numbers of preimages of points are uniformly bounded. A point $x \in X$ is said to be *singular* (for T) if at least one of the following two conditions is satisfied:

(i) There exists no open neighborhood U of x on which T is injective.
(ii) For each open neighborhood U of x, there exists an open set $V \subseteq U$ whose image $T(V)$ is not open.

Denote by $\mathrm{Sing}(T)$ the set of all singular points for the map T.

To establish Theorem 1.3, we need some technical results. First, we formulate the existence of Jacobians and Rokhlin's entropy formula in our context as follows.

Proposition 5.3. *Let (X,ρ) be a compact metric space, $T\colon X\to X$ a finite-to-one continuous map with finitely many singular points and finite topological entropy, and $\mu\in\mathcal{P}(X)$ a Borel probability measure on X. Then there exists a Jacobian $J_\mu\colon X\to[0,+\infty)$ for T with respect to μ. Moreover, if we assume that μ is a T-invariant Borel probability measure on X, then we have*

$$h_\mu(T)=\int\log(J_\mu)\,\mathrm{d}\mu. \tag{5.2}$$

Under the conditions of Proposition 5.3, the existence of Jacobians follows immediately from [PU10, Proposition 2.9.5], and (5.2) follows immediately from [PU10, Theorems 2.9.7 and 2.9.8].

The following proposition gives the definitions and properties of transfer operators. For our purpose, we state it in the following form and include the proof for the sake of completeness.

Proposition 5.4. *Let (X,ρ) be a compact metric space, $T\colon X\to X$ a finite-to-one continuous map with finitely many singular points. Assume that μ is a T-invariant Borel probability measure on X, J is a Jacobian for T with respect to μ. Let $L^1(\mu)$ denote the space of L^1 functions $u\colon X\to\mathbb{R}$ with respect to the measure μ. Then the following statements hold:*

(i) *There exists a sequence $\{X_j\}_{j\in\mathbb{N}}$ of pairwise disjoint Borel sets satisfying that $X=\bigcup_{j\in\mathbb{N}}X_j$ and $T|_{X_j}$ is a homeomorphism of X_j onto $T(X_j)$ for each $j\in\mathbb{N}$.*

(ii) *There exists a sequence $\{\Phi_j\}_{j\in\mathbb{N}}$ of μ-integrable and non-negative functions on X such that*

$$\mu\bigl(T^{-1}(B)\cap X_j\bigr)=\int_B\Phi_j\,\mathrm{d}\mu\quad\text{for each } \mu\text{-measurable set } B\subseteq X \text{ and each } j\in\mathbb{N}. \tag{5.3}$$

(iii) *Define Ψ on X by*

$$\Psi(x)\coloneqq\Phi_j(T(x))\quad\text{for each } j\in\mathbb{N} \text{ and each } x\in X_j. \tag{5.4}$$

Then the following properties hold:

(a) *J is a Jacobian on $X\smallsetminus\Psi^{-1}(0)$ for T with respect to μ.*

(b) *For μ-a.e. $x\in X$, if there exists $j\in\mathbb{N}$ with $x\in T(X_j)$, then*

$$\Phi_j(x)\cdot J\bigl((T|_{X_j})^{-1}(x)\bigr)=\begin{cases}0 & \text{if } \Phi_j(x)=0;\\ 1 & \text{if } \Phi_j(x)\neq 0.\end{cases} \tag{5.5}$$

(iv) *Define $\mathcal{L}_\mu\colon L^1(\mu)\to L^1(\mu)$ by*

$$\mathcal{L}_\mu(u)(x)\coloneqq\sum_{y\in T^{-1}(x)}u(y)\Psi(y)\quad\text{for all } u\in L^1(\mu) \text{ and } x\in X. \tag{5.6}$$

Then $\mathcal{L}_\mu$ is a well-defined operator on $L^1(\mu)$. Moreover, we have

$$\int\mathcal{L}_\mu(u)\,\mathrm{d}\mu=\int u\,\mathrm{d}\mu\quad\text{for each } u\in L^1(\mu)\quad\text{and} \tag{5.7}$$

$$\mathcal{L}_\mu(\mathbb{1})(x)=1\quad\text{for } \mu\text{-a.e. } x\in X. \tag{5.8}$$

Proof. (i) By Definition 5.2, for each $x\in X\smallsetminus\operatorname{Sing}(T)$, there exists an open neighborhood U_x of x that satisfies that T is injective and open on U_x and $U_x\cap\operatorname{Sing}(T)=\emptyset$. Since T is also continuous on X, $T|_{U_x}$ is a homeomorphism of U_x onto $T(U_x)$. On the other hand, since (X,ρ) is a compact metric space, there exists a countable basis of X for its topology, say $\{V_i\}_{i\in\mathbb{N}}$. Since $\operatorname{Sing}(T)$ is a finite close set, $\{V_i\smallsetminus\operatorname{Sing}(T)\}_{i\in\mathbb{N}}$ is a countable basis of $X\smallsetminus\operatorname{Sing}(T)$ for its topology. Note that $\{U_x : x\in X\smallsetminus\operatorname{Sing}(T)\}$ is an open covering of $X\smallsetminus\operatorname{Sing}(T)$. Then by the Lindelöf Covering Theorem (see for example, [Mun00, Theorem 30.3 (a)]), there exists a sequence $\{x_j\}_{j\in\mathbb{N}}$ of points in X such that $X\smallsetminus\operatorname{Sing}(T)=\bigcup_{j\in\mathbb{N}}U_{x_j}$.

Let N denote $\operatorname{card}(\operatorname{Sing}(T))$. Now we define $\{X_j\}_{j\in\mathbb{N}}$ as follows:

$$X_j\coloneqq\begin{cases}\{m_j\} & \text{if } 1\leqslant j\leqslant N;\\ U_{x_{j-N}}\smallsetminus\bigcup_{k=1}^{j-1-N}U_{x_k} & \text{if } j\geqslant N+1\end{cases}$$

for each $j \in \mathbb{N}$. Here $\operatorname{Sing}(T) = \{m_j\}_{j=1}^N$. Obviously, the sequence $\{X_j\}_{j\in\mathbb{N}}$ satisfies all requirements in statement (i). Therefore, statement (i) follows.

(ii) Fix an arbitrary $j \in \mathbb{N}$. By statement (i), $\bigl(T|_{X_j}\bigr)^{-1} \colon T(X_j) \to X_j$ is a homeomorphism. Let μ_j denote the restriction of μ on the Borel set $T(X_j)$, and $\widetilde{\mu}_j$ a function defined on the set of Borel subsets of $T(X_j)$ in such a way that $0 \leqslant \widetilde{\mu}_j(B) \coloneqq \mu\bigl(\bigl(T|_{X_j}\bigr)^{-1}(B)\bigr)$ for each Borel $B \subseteq T(X_j)$. It is clear to see that μ_j and $\widetilde{\mu}_j$ are positive Borel measures on $T(X_j)$. On the other hand, by the T-invariance of the measure μ, we have $\mu_j(B) = 0$ implies that $\widetilde{\mu}_j(B) \leqslant \mu\bigl(T^{-1}(B)\bigr) = \mu(B) = \mu_j(B) = 0$ for each Borel $B \subseteq T(X_j)$, i.e., $\tilde{\mu}_j$ is absolutely continuous with respect to the measure μ_j. Define Φ_j as follows:

$$\Phi_j(x) \coloneqq \begin{cases} \frac{\mathrm{d}\widetilde{\mu}_j}{\mathrm{d}\mu_j}(x) & \text{if } x \in T(X_j); \\ 0 & \text{if } x \notin T(X_j) \end{cases}$$

for each $x \in X$. Note that μ_j and $\tilde{\mu}_j$ are both positive measures. Then the Radon–Nikodym derivative $\frac{\mathrm{d}\widetilde{\mu}_j}{\mathrm{d}\mu_j}$ is a μ_j-integrable and non-negative function. Since $T(X_j)$ is a Borel set, Φ_j is μ-integrable and (5.3) holds.

(iii) We first prove that $\mu\bigl(\Psi^{-1}(0)\bigr) = 0$. By (5.3) and (5.4), we have

$$\mu\bigl(\Psi^{-1}(0) \cap X_j\bigr) = \mu\bigl(T^{-1}\bigl(\Phi_j^{-1}(0)\bigr) \cap X_j\bigr) = \int_{\Phi_j^{-1}(0)} \Phi_j \,\mathrm{d}\mu = 0 \quad \text{for each } j \in \mathbb{N}.$$

Note that $X = \bigcup_{j\in\mathbb{N}} X_j$. Thus $\mu\bigl(\Psi^{-1}(0)\bigr) = 0$. By Definition 5.1, there exists a Borel set E with full μ-measure satisfying that J is a Jacobian on E with respect to the measure μ. Since $X = \bigcup_{j\in\mathbb{N}} X_j$ and T is injective on X_j for each $j \in \mathbb{N}$. In order to prove statement (iii) (a), it suffices to show that (5.1) is true for each $j \in \mathbb{N}$ and each Borel subset $A \subseteq X_j \smallsetminus \bigl(\Psi^{-1}(0) \cup E\bigr)$.

Now fix an integer j and a Borel subset $A \subseteq X_j \smallsetminus \bigl(\Psi^{-1}(0) \cup E\bigr)$. Since $\mu(E) = 1$, we have $0 \leqslant \mu(A) \leqslant 1 - \mu(E) = 0$. Note that $A \subseteq X_j \smallsetminus \Psi^{-1}(0)$ and T is injective on X_j, then $A = T^{-1}(T(A)) \cap X_j$ and $\Phi_j(T(x)) = \Psi(x) > 0$ for each $x \in A$. On the other hand, by (5.3), $0 = \mu(A) = \mu\bigl(T^{-1}(T(A)) \cap X_j\bigr) = \int_{T(A)} \Phi_j \,\mathrm{d}\mu$. Note that by $A \subseteq X_j \smallsetminus \bigl(\Psi^{-1}(0) \cup E\bigr)$, $\Phi_j(y) > 0$ for each $y \in T(A)$. Then it follows from $\int_{T(A)} \Phi_j \,\mathrm{d}\mu = 0$ that $\mu(T(A)) = 0$. Therefore, (5.1) is true for the set A.

Then we prove statement (iii) (b). Fix an arbitrary $j \in \mathbb{N}$. Consider the map

$$T_j \coloneqq T|_{X_j \smallsetminus T^{-1}(\Phi_j^{-1}(0))} \colon X_j \smallsetminus T^{-1}\bigl(\Phi_j^{-1}(0)\bigr) \to T(X_j) \smallsetminus \Phi_j^{-1}(0).$$

Then T_j is a homeomorphism. Since J is a Jacobian on $X \smallsetminus \Psi^{-1}(0)$, by Definition 5.1, we have for each Borel $A \subseteq X$ satisfying that $T_j^{-1}(A) \subseteq X_j \smallsetminus T^{-1}\bigl(\Phi_j^{-1}(0)\bigr) \subseteq X \smallsetminus \Psi^{-1}(0)$,

$$\mu(A) = \int_{T_j^{-1}(A)} J \,\mathrm{d}\mu. \tag{5.9}$$

By (5.3), we have

$$\mu\bigl(T_j^{-1}(A)\bigr) = \int_A \Phi_j \,\mathrm{d}\mu \tag{5.10}$$

for each Borel $A \subseteq T(X_j) \smallsetminus \Phi_j^{-1}(0)$. By the change of variable formula, it is not hard to derive from (5.9) and (5.10) that for each $j \in \mathbb{N}$ and μ-a.e. $x \in X$, if $x \in T(X_j) \smallsetminus \Phi_j^{-1}(0)$, then $\Phi_j(x) \cdot J\bigl((T|_{X_j})^{-1}(x)\bigr) = 1$; if $x \in \Phi_j^{-1}(0)$, then $\Phi_j(x) \cdot J\bigl((T|_{X_j})^{-1}(x)\bigr) = 0$. Therefore, we obtain (5.5).

(iv) Note that by (5.6), $\mathcal{L}_\mu$ is a linear opertor. Thus, in order to show that $\mathcal{L}_\mu$ is a well-defined operator on $L^1(\mu)$, it suffices to prove that for each function $u \in L^1(\mu)$ that equals zero except on a set C of zero μ-measure, $\mathcal{L}_\mu(u)(x) = 0$ for μ-a.e. $x \in X$. Since $X = \bigcup_{j\in\mathbb{N}} X_j$, without loss of generality, we assume that $C \subseteq X_j$ for some $j \in \mathbb{N}$. Then by (5.3), we obtain that

$$0 = \mu(C) = \mu\bigl(T^{-1}(T(C)) \cap X_j\bigr) = \int_{T(C)} \Phi_j \,\mathrm{d}\mu.$$

Note that Φ_j is a non-negative function. Then $\mu\bigl(\{x \in T(C) : \Phi_j(x) > 0\}\bigr) = 0$. Thus, it follows from (5.6) that $\mathcal{L}_\mu(u)(x) = 0$ for μ-a.e. $x \in X$. Therefore, $\mathcal{L}_\mu$ is a well-defined operator on $L^1(\mu)$.

We now prove (5.7). Indeed, it is suffices to show that (5.7) is true for each indicator function $\mathbb{1}_D$ and each μ-measurable subset D contained in X_j for some $j \in \mathbb{N}$. By (5.6), (5.4), and (5.3), we have

$$\int \mathcal{L}_\mu(\mathbb{1}_D)\,\mathrm{d}\mu = \int_{T(D)} \Psi \circ (T|_{X_j})^{-1}\,\mathrm{d}\mu = \int_{T(D)} \Phi_j\,\mathrm{d}\mu = \mu(D)$$

for each μ-measurable subset D contained in X_j for some $j \in \mathbb{N}$.

Finally, we establish (5.8). By (5.3), we obtain that for each $j \in \mathbb{N}$ and μ-a.e. $x \in X$, if $x \notin T(X_j)$, $\Phi_j(x) = 0$. Combining with (5.6) and (5.4), we have

$$\mathcal{L}_\mu(\mathbb{1})(x) = \sum_{y \in T^{-1}(x)} \Psi(y) = \sum_{j \in \mathbb{N}: T^{-1}(x) \cap X_j \neq \emptyset} \Phi_j(x) = \sum_{j \in \mathbb{N}} \Phi_j(x) \quad \text{for } \mu\text{-a.e. } x \in X. \tag{5.11}$$

Then by (5.11) and (5.3), we obtain that for each μ-measurable set $A \subseteq X$,

$$\int_A \mathcal{L}_\mu(\mathbb{1})\,\mathrm{d}\mu = \sum_{j \in \mathbb{N}} \int_A \Phi_j\,\mathrm{d}\mu = \sum_{j \in \mathbb{N}} \mu\bigl(T^{-1}(A) \cap X_j\bigr) = \mu\bigl(T^{-1}(A)\bigr) = \mu(A),$$

where the third quality follows from the fact that X is the disjoint union of $\{X_j\}_{j \in \mathbb{N}}$, and the last quality follows from the T-invariance of the measure μ. Therefore, we obtain (5.8). ☐

As an application of Proposition 5.4, we obtain a description of Jacobians on some Borel subsets with full μ-measure for finite-to-one continuous maps T with respect to T-invariant measures μ on X.

Theorem 5.5. *Let (X, ρ) be a compact metric space, $T \colon X \to X$ a finite-to-one continuous map with finitely many singular points and finite topological entropy, and μ a T-invariant Borel probability measure on X. Assume that $J \colon X \to \mathbb{R}^+$ is a μ-measurable function and $Y \subseteq X$ is a subset with $\mu(Y) = 1$. Then J is a Jacobian on Y for the map T with respect to the measure μ if and only if the following equations are satisfied:*

$$h_\mu(T) = \int \log(J)\,\mathrm{d}\mu \quad \textit{and} \tag{5.12}$$

$$\sum_{y \in T^{-1}(x) \cap Y} \frac{1}{J(y)} = 1 \quad \textit{for } \mu\textit{-a.e. } x \in X. \tag{5.13}$$

Proof. By Proposition 5.4 (i), there exists a countable set $\{X_j\}_{j \in \mathbb{N}}$ that consists of pairwise disjoint Borel subsets of X satisfying that $X = \bigcup_{j \in \mathbb{N}} X_j$, and $T|_{X_j}$ is a homeomorphism of X_j onto $T(X_j)$ for each $j \in \mathbb{N}$. By Proposition 5.4 (ii), there exists a sequence $\{\Phi_j\}_{j \in \mathbb{N}}$ of μ-integrable and non-negative functions on X satisfying (5.3). Let the function Ψ and the operator $\mathcal{L}_\mu \colon L^1(\mu) \to L^1(\mu)$ be defined as in Proposition 5.4 (iii) and (iv), respectively.

Denote by U_μ the set $X \smallsetminus \Psi^{-1}(0)$. For each pair of subsets A, B of X, denote by $A \triangle B$ the set $(A \smallsetminus B) \cup (B \smallsetminus A)$.

Next, we establish the forward implication in Theorem 5.5. Assume that J is a Jacobian on Y for the map T with respect to the measure μ, where $\mu(Y) = 1$. Then (5.12) follows immediately from Proposition 5.3. Hence, to prove the forward implication, it suffices to show that the function J satisfies (5.13). Note that for each $x \notin T(U_\mu \triangle Y)$, $T^{-1}(x) \cap U_\mu = T^{-1}(x) \cap Y$. Then we obtain that

$$\sum_{y \in T^{-1}(x) \cap Y} \frac{1}{J(y)} = \sum_{y \in T^{-1}(x) \cap U_\mu} \frac{1}{J(y)} \quad \text{for each } x \in X \smallsetminus T(U_\mu \triangle Y). \tag{5.14}$$

Now we claim that $\mu(T(U_\mu \triangle Y)) = 0$. By Proposition 5.4 (iii) (a), J is a Jacobian on $U_\mu = X \smallsetminus \Psi^{-1}(0)$. Hence, by (5.1) and Proposition 5.4 (i), it is not hard to derive from $\mu(U_\mu \smallsetminus Y) = 0$ that $\mu(T(U_\mu \smallsetminus Y)) = 0$. Since J is a Jacobian on Y, by (5.1) and Proposition 5.4 (i), similarly, it is not hard to derive from $\mu(Y \smallsetminus U_\mu) = 0$ that $\mu(T(Y \smallsetminus U_\mu)) = 0$. Thus, we obtain that

$$0 \leqslant \mu(T(U_\mu \triangle Y)) \leqslant \mu(T(U_\mu \smallsetminus Y)) + \mu(T(Y \smallsetminus U_\mu)) = 0,$$

hence, $\mu(T(U_\mu \triangle Y)) = 0$. Since $U_\mu = X \smallsetminus \Psi^{-1}(0)$, $\Phi_j(T(y)) = \Psi(y) \neq 0$ for each $j \in \mathbb{N}$ and each $y \in U_\mu \cap X_j$. Then by Proposition 5.4 (iii) (b), we have for μ-a.e. $x \in X$, $\Psi(y) = \frac{1}{J(y)}$ for each $y \in T^{-1}(x) \cap U_\mu$. Hence, noting that $(U_\mu)^c = \Psi^{-1}(0)$, by (5.14), (5.6), and 5.8, for μ-a.e. $x \in X$,

$$\sum_{y \in T^{-1}(x) \cap Y} \frac{1}{J(y)} = \sum_{y \in T^{-1}(x) \cap U_\mu} \frac{1}{J(y)} = \sum_{y \in T^{-1}(x) \cap U_\mu} \Psi(y) = \sum_{y \in T^{-1}(x)} \Psi(y) = \mathcal{L}_\mu(\mathbb{1})(x) = 1.$$

Hence, (5.13) holds for the function J.

To prove the backward implication of Theorem 5.5, we assume that J satisfies (5.12) and (5.13), and show that J is a Jacobian on Y for the map T with respect to the measure μ. For each $x \in X \smallsetminus T(U_\mu \smallsetminus Y)$, since $T^{-1}(x) \cap U_\mu \subseteq T^{-1}(x) \cap Y$, we have

$$\sum_{y \in T^{-1}(x) \cap U_\mu} \frac{1}{J(y)} \leqslant \sum_{y \in T^{-1}(x) \cap Y} \frac{1}{J(y)}. \tag{5.15}$$

Since $\mu(Y) = 1$, we have $\mu(U_\mu \smallsetminus Y) = 0$. Note that, by Proposition 5.4 (iii) (a), J is a Jacobian on U_μ. By (5.1) and Proposition 5.4 (i), it is not hard to see that $\mu(T(U_\mu \smallsetminus Y)) = 0$. Hence, (5.15) holds for μ-a.e. $x \in X$. By Proposition 5.3, there exists a Jacobian J_μ for the map T with respect to the measure μ. Similarly, noting that $(U_\mu)^c = \Psi^{-1}(0)$, by (5.6), (5.5), (5.15), and (5.13), it follows from $U_\mu = X \smallsetminus \Psi^{-1}(0)$ that for μ-a.e. $x \in X$,

$$\begin{aligned} \mathcal{L}_\mu\big(J_\mu/J\big)(x) &= \sum_{y \in T^{-1}(x)} \frac{J_\mu(y)\Psi(y)}{J(y)} = \sum_{y \in T^{-1}(x) \cap U_\mu} \frac{J_\mu(y)\Psi(y)}{J(y)} \\ &= \sum_{y \in T^{-1}(x) \cap U_\mu} \frac{1}{J(y)} \leqslant \sum_{y \in T^{-1}(x) \cap Y} \frac{1}{J(y)} = 1. \end{aligned} \tag{5.16}$$

Hence, by (5.16), (5.7), (5.12), and Proposition 5.3, we obtain that

$$1 = \int \mathbb{1} \, \mathrm{d}\mu \geqslant \int \mathcal{L}_\mu(J_\mu/J) \, \mathrm{d}\mu = \int \frac{J_\mu}{J} \, \mathrm{d}\mu \geqslant 1 - \int \log(J) \, \mathrm{d}\mu + \int \log(J_\mu) \, \mathrm{d}\mu = 1. \tag{5.17}$$

The last inequality holds since $1 + \log(x) \leqslant x$ for each $x > 0$, where the equality holds if and only if $x = 1$. Thus, all the inequalities in (5.17) must be equalities. Thus we obtain that $J_\mu(x) = J(x)$, and $\mathcal{L}_\mu(J_\mu/J)(x) = 1$ for μ-a.e. $x \in X$. Hence, the inequality in (5.16) must be an equality, which implies that $T^{-1}(x) \cap U_\mu = T^{-1}(x) \cap Y$ for μ-a.e. $x \in X$. Note that for each $x \in T(Y \smallsetminus U_\mu)$, $T^{-1}(x) \cap U_\mu \neq T^{-1}(x) \cap Y$. Then we obtain that $\mu(T(Y \smallsetminus U_\mu)) = 0$. Since J is a Jacobian on U_μ, it follows from $\mu(T(Y \smallsetminus U_\mu)) = 0$ that $J_\mu = J$ is a Jacobian on Y. □

By Theorem 5.5 and the definition of equilibrium states, we obtain the following corollary.

Corollary 5.6. *Let (X, ρ) be a compact metric space, $T\colon X \to X$ a finite-to-one continuous map with finitely many singular points and finite topological entropy, $\phi\colon X \to \mathbb{R}$ a continuous function, and C a Borel subset of X. Assume that J is a Borel measurable positive function that satisfies the following properties:*

(i) *For each $x \in T(C \smallsetminus \mathrm{Sing}(T))$,*

$$\sum_{y \in T^{-1}(x) \cap (C \smallsetminus \mathrm{Sing}(T))} \frac{1}{J(y)} = 1.$$

(ii) *There exists a continuous function $h\colon X \to \mathbb{R}$ that satisfies that*

$$J(x) = \exp(P(T, \phi) - \phi(x) + h(T(x)) - h(x)) \quad \textit{for each } x \in C \smallsetminus \mathrm{Sing}(T).$$

Then for each $\mu \in \mathcal{M}(X, T) \cap \mathcal{P}(X, C \smallsetminus \mathrm{Sing}(T))$, μ is an equilibrium state for the map T and the potential ϕ if and only if J is a Jacobian on $C \smallsetminus \mathrm{Sing}(T)$ for T with respect to μ.

Proof. Fix an arbitrary $\mu \in \mathcal{M}(X,T) \cap \mathcal{P}(X, C \smallsetminus \mathrm{Sing}(T))$. Then $\mu(C \smallsetminus \mathrm{Sing}(T)) = 1$. Hence, by property (ii) of J, we obtain that

$$\int \log(J)\,\mathrm{d}\mu = P(T,\phi) - \int \phi\,\mathrm{d}\mu + \int h \circ T\,\mathrm{d}\mu - \int h\,\mathrm{d}\mu = P(T,\phi) - \int \phi\,\mathrm{d}\mu. \tag{5.18}$$

Assume that μ is an equilibrium state for T and ϕ. Then by (5.18), we have $h_\mu(T) = P(T,\phi) - \int \phi\,\mathrm{d}\mu = \int \log(J)\,\mathrm{d}\mu$. Combining with property (i) of J and Theorem 5.5, this implies that J is a Jacobian on $C \smallsetminus \mathrm{Sing}(T)$ for T with respect to μ.

For the opposite direction, we assume that J is a Jacobian on $C \smallsetminus \mathrm{Sing}(T)$ for T with respect to μ. By (5.12) in Theorem 5.5 and (5.18), we have $h_\mu(T) = \int \log(J)\,\mathrm{d}\mu = P(T,\phi) - \int \phi\,\mathrm{d}\mu$, i.e., μ is an equilibrium state for the map T and potential ϕ. □

5.2. **Proof of Theorem 1.3.** In this subsection, we establish Lemma 5.7 and use it to prove Proposition 5.8 first. Then we establish Lemma 5.10 on the effective openness of maps as preparation. Finally, to close this subsection, we complete the proof of Theorem 1.3.

Lemma 5.7. *Let $(X, \rho, \mathcal{S})$ be a computable metric space and X be a recursively compact set. Then there exists an algorithm $\mathcal{A}^0(\cdot,\cdot,\cdot)$ satisfying the following property:*

For each $r \in \mathbb{R}^+$, each recursively compact subset $K \subseteq X$, and each lower semi-computable open set $U \subseteq X$, the algorithm $\mathcal{A}^0(r,K,U)$ halts if and only if $\rho(K,U^c) > r$ after inputting the following data in this algorithm:

(i) *an algorithm computing the positive number r,*
(ii) *an algorithm computing a function $f \colon \mathbb{N} \to \mathbb{N}$ satisfying that $U = \bigcup_{n\in\mathbb{N}} B_{f(n)}$, and*
(iii) *an algorithm which, on input a sequence $\{k_j\}_{j=1}^p$ of integers and a sequence $\{r_j\}_{j=1}^p$ of positive rational numbers, halts if and only if $K \subseteq \bigcup_{j=1}^p B\big(s_{k_j}, r_j\big)$.*

Proof. Fix an arbitrary recursively compact subset $K \subseteq X$ and an arbitrary lower semi-computable open subset $U \subseteq X$. Since X is a recursively compact set, it follows from Proposition 3.18 (ii) that $U^c = X \smallsetminus U$ is a recursively compact set. Since $(X, \rho, \mathcal{S})$ is a computable metric space, by Definition 3.3, it is not hard to see that $(X \times X, \widehat{\rho}, \mathcal{S} \times \mathcal{S})$ is a computable metric space if we define $\widehat{\rho}((x,y),(x',y')) \coloneqq \max\{\rho(x,x'), \rho(y,y')\}$.

Claim. $K \times U^c$ is a recursively compact set in $(X \times X, \widehat{\rho}, \mathcal{S} \times \mathcal{S})$.

Indeed, since X is recursively compact, by Proposition 3.16, $(X, \rho, \mathcal{S})$ is recursively precompact and (X,ρ) is complete. Since $(X, \rho, \mathcal{S})$ is recursively precompact, it is not hard to derive from Definition 3.15 that $(X \times X, \widehat{\rho}, \mathcal{S} \times \mathcal{S})$ is recursively precompact. Since (X,ρ) is complete, $(X \times X, \widehat{\rho})$ is complete. Thus, by Proposition 3.16, we obtain that $X \times X$ is recursively compact. On the other hand, since U is a lower semi-computable open set, by Definition 3.6, it is not hard to see that $X \times U$ is also a lower semi-computable open set. Since K is a recursively compact set, by Proposition 3.18 (v), $K^c = X \smallsetminus K$ is a lower semi-computable open set. Similarly, we obtain that $K^c \times X$ is also a lower semi-computable open set. Since $X \times X$ is a recursively compact set, by Proposition 3.18 (ii), $K \times X = (X \times X) \smallsetminus (K^c \times X)$ and $X \times U^c = (X \times X) \smallsetminus (X \times U)$ are both recursively compact sets. Hence, by Proposition 3.18 (vi), $K \times U^c = (K \times X) \cap (X \times U^c)$ is a recursively compact set, establishing the claim.

Since, by Proposition 3.18 (iv), $\rho(K,U^c) = \inf_{(x,y)\in K\times U^c} \rho(x,y)$ is lower semi-computable. This implies that, by Definition 3.2 (iii), there is an algorithm $\mathcal{A}(\cdot,\cdot)$ that satisfies the following result. If we input data (ii) and (iii) in the algorithm, then $\mathcal{A}(K,U)$ outputs a sequence $\{r_i\}_{i\in\mathbb{N}}$ of rational numbers satisfying that $\{r_i\}_{i\in\mathbb{N}}$ is strictly increasing and converges to $\rho(K,U^c)$ for each recursively compact $K \subseteq X$ and lower semi-computable open $U \subseteq X$. Thus, by checking whether there exists $i \in \mathbb{N}$ such that $r_i > r$ with data (i), we can check whether $\rho(K,U^c) > r$ holds. Hence, this gives the algorithm $\mathcal{A}^0(\cdot,\cdot,\cdot)$ with desired property. □

As a convention, for each pair of subsets A and B of a given linear space, define the linear convex hull of A and B as follows:

$$\mathrm{cl}(A,B) \coloneqq \{\lambda a + (1-\lambda)b \colon \lambda \in [0,1],\ a \in A, \text{ and } b \in B\}.$$

Then we can introduce the following result.

Proposition 5.8. *Let $(X,\,\rho,\,\mathcal{S})$ be a computable metric space and X be a recursively compact set. Assume that K is a compact subset of $\mathcal{P}(X)$, and μ_0 is a computable point in $\mathcal{P}(X)$. Then the following statements hold:*

(i) *If K is recursively compact, then $\operatorname{cl}(K, \{\mu_0\})$ is also recursively compact.*

(ii) *Assume that $\operatorname{cl}(K, \{\mu_0\})$ is a recursively compact set, and that for all $\nu \in \mathcal{P}(X)$ and $\lambda \in [0,1]$, if $\lambda\nu + (1-\lambda)\mu_0 \in K$, then $\lambda = 1$. Then K is also a recursively compact set.*

Proof. (i) Since $(X,\,\rho,\,\mathcal{S})$ is a computable metric space and X is a recursively compact set, it follows from Proposition 3.20 that $(\mathcal{P}(X),\,W_\rho,\,\mathcal{Q}_\mathcal{S})$ is also a computable metric space and $\mathcal{P}(X)$ is also a recursively compact set. Since μ_0 is a computable point and K is a recursively compact set, by Proposition 3.18 (iv), $\sup_{\mu\in K} W_\rho(\mu,\mu_0)$ is upper semi-computable. Hence, by Definition 3.2, we can compute a strictly decreasing sequence $\{r_i\}_{i\in\mathbb{N}}$ of uniformly computable rational numbers converging to $\sup_{\mu\in K} W_\rho(\mu,\mu_0)$.

Denote by K_λ the set $\{\lambda\mu + (1-\lambda)\mu_0 \colon \mu \in K\}$ for each $\lambda \in [0,1]$. Note that for each sequence $\bigl\{B_{W_\rho}\bigl(\mu_j, q_j\bigr)\bigr\}_{j=1}^p$ of ideal balls in $\mathcal{P}(X)$, $K_\lambda \subseteq \bigcup_{j=1}^p B_{W_\rho}(\mu_j, q_j)$ if and only if $K \subseteq \bigcup_{j=1}^p B_{W_\rho}\bigl(\frac{\mu_j-\mu_0}{\lambda} + \mu_0, \frac{q_j}{\lambda}\bigr)$. Combining with the fact that K is recursively compact and Definition 3.14, this implies that for each rational number $0 \leqslant \lambda \leqslant 1$, K_λ is a recursively compact set.

Let $\mathcal{A}_K^1(\cdot,\cdot,\cdot)$ be an algorithm that satisfies the following property. For all $n \in \mathbb{N}$, $\delta \in \mathbb{Q}^+$, and lower semi-computable open set $U \subseteq \mathcal{P}(X)$, the algorithm $\mathcal{A}_K^1(n, \delta_0, U)$ halts if and only if $\mathcal{A}^0(\delta_0,\, K_{i/n},\, U)$ halts for each $0 \leqslant i \leqslant n$, after inputting n, δ, and an algorithm computing a function $f \colon \mathbb{N} \to \mathbb{N}$ satisfying that $U = \bigcup_{n\in\mathbb{N}} B_{f(n)}$. Here $\{B_i\}_{i\in\mathbb{N}}$ is the effective enumeration of ideal balls in $(\mathcal{P}(X),\, W_\rho,\, \mathcal{Q}_\mathcal{S})$.

In the following, we establish an algorithm $\mathcal{A}^2(\cdot,\cdot)$ such that for each sequence $\{i_j\}_{j=1}^p$ of positive integers and each sequence $\{q_j\}_{j=1}^p$ of positive rational numbers, the algorithm $\mathcal{A}^2\bigl(\{i_j\}_{j=1}^p, \{q_j\}_{j=1}^p\bigr)$ halts if and only if $K \subseteq \bigcup_{j=1}^p B_{W_\rho}\bigl(s_{i_j}, q_j\bigr)$.

Begin

(i) Read the sequences $\{i_j\}_{j=1}^p$ and $\{q_j\}_{j=1}^p$.

(ii) Compute a strictly decreasing sequence $\{r_i\}_{i\in\mathbb{N}}$ of uniformly computable rational numbers converging to $\sup_{\mu\in K} W_\rho(\mu,\mu_0)$. Set $r = r_1$, then $r > W_\rho(\mu,\mu_0)$ for each $\mu \in K$.

(iii) Set u to be 1.

(iv) **While** $u \geqslant 1$ **do**

  (1) Run the algorithm $\mathcal{A}_K^1\bigl(u,\, r/u,\, \bigcup_{j=1}^p B_{W_\rho}\bigl(s_{i_j}, q_j\bigr)\bigr)$.

  (2) Set v to be 1.

  (3) **While** $1 \leqslant v \leqslant u$ **do**

    (a) If $\mathcal{A}_K^1\bigl(v,\, r/v,\, \bigcup_{j=1}^p B_{W_\rho}\bigl(s_{i_j}, q_j\bigr)\bigr)$ halts, then set u to be -1.

    (b) Set v to be $v+1$.

  (4) Set u to $u+1$.

End

Now we verify the feasibility of the above algorithm. We fix sequences $\{i_j\}_{j=1}^p$ and $\{q_j\}_{j=1}^p$ and show that $\mathcal{A}^2\bigl(\{i_j\}_{j=1}^p, \{q_j\}_{j=1}^p\bigr)$ halts if and only if $K \subseteq V$, where $V \coloneqq \bigcup_{j=1}^p B_{W_\rho}\bigl(s_{i_j}, q_j\bigr)$.

First, we assume that the algorithm $\mathcal{A}^2\bigl(\{i_j\}_{j=1}^p, \{q_j\}_{j=1}^p\bigr)$ halts. Then there exists $v \in \mathbb{N}$ satisfying that $\mathcal{A}_K^1(v,\, r/v,\, V)$ halts (see Step (iv) (3) (a)), where $r \in \mathbb{Q}^+$ satisfies that $r > W_\rho(\mu,\mu_0)$ for each $\mu \in K$ (see Step (ii)). Hence, by Lemma 5.7, we have $r/v < W_\rho(K_{i/v}, V^c)$ for each integer $0 \leqslant i \leqslant v$.

Fix an arbitrary integer $1 \leqslant i \leqslant v-1$. For each $\lambda \in \bigl[\frac{2i-1}{2v}, \frac{2i+1}{2v}\bigr]$ and $\mu \in K$, by (2.1), we obtain that

$$\begin{aligned} W_\rho\Bigl(\lambda\mu + (1-\lambda)\mu_0, \frac{i\mu + (v-i)\mu_0}{v}\Bigr) &= \sup\Bigl\{\Bigl|\Bigl(\lambda - \frac{i}{v}\Bigr)(\langle\mu, f\rangle - \langle\mu_0, f\rangle)\Bigr| : f \in C^{0,1}(X,d), \lvert f\rvert_{1,d} \leqslant 1\Bigr\} \\ &\leqslant \Bigl|\lambda - \frac{i}{v}\Bigr| \cdot W_\rho(\mu,\mu_0) \leqslant \frac{r}{2v} < \frac{r}{v}. \end{aligned}$$

Thus, for each integer $1 \leqslant i \leqslant v-1$,

$$\bigcup_{\lambda\in\left[\frac{2i-1}{2v},\frac{2i+1}{2v}\right]} K_\lambda \subseteq B_{W_\rho}\bigl(K_{i/v}, r/v\bigr). \tag{5.19}$$

Similarly, we obtain that

$$\bigcup_{\lambda\in\left[0,\frac{1}{2v}\right]} K_\lambda \subseteq B_{W_\rho}(K_0, r/v) \quad \text{and} \quad \bigcup_{\lambda\in\left[\frac{2v-1}{2v},1\right]} K_\lambda \subseteq B_{W_\rho}(K_1, r/v). \tag{5.20}$$

Since $\mathrm{cl}(K,\{\mu_0\}) = \bigcup_{\lambda\in[0,1]} K_\lambda$ and $r/v < W_\rho(K_{i/v}, V^c)$ for each integer $0 \leqslant i \leqslant v$, it follows from (5.19) and (5.20) that

$$\mathrm{cl}(K,\{\mu_0\}) \subseteq B_{W_\rho}(K_0, r/v) \cup \left(\bigcup_{i=1}^{v-1} B_{W_\rho}(K_{i/v}, r/v)\right) \cup B_{W_\rho}(K_1, r/v) \subseteq V.$$

For the opposite direction, we assume that $\mathrm{cl}(K,\{\mu_0\}) \subseteq V$ and r is the rational number that is computed in Step (ii) in $\mathcal{A}^2\bigl(\{i_j\}_{j=1}^p, \{q_j\}_{j=1}^p\bigr)$. Set $n_1 := \bigl[\frac{r}{W_\rho(\mathrm{cl}(K,\{\mu_0\}),V^c)}\bigr] + 1$. Then it is not hard to see that $\frac{r}{n_1} < W_\rho(\mathrm{cl}(K,\{\mu_0\}), V^c)$, i.e., $\mathcal{A}^0(r/n_1,\, K_{i/n_1},\, V^c)$ halts eventually for each integer $0 \leqslant i \leqslant n_1$. Thus, by definition of the algorithm $\mathcal{A}^1_K$, $\mathcal{A}^1_K(n_1,\, r/n_1,\, V)$ halts eventually. Hence, the algorithm $\mathcal{A}^2\bigl(\{i_j\}_{j=1}^p, \{q_j\}_{j=1}^p\bigr)$ halts eventually. Therefore, by Definition 3.14, $\mathrm{cl}(K,\{\mu_0\})$ is recursively compact.

(ii) We first establish the following claim. For each subset $U \subseteq \mathcal{P}(X)$, denote by $\mathrm{ol}(U,\{\mu_0\})$ the set $\{\lambda\mu + (1-\lambda)\mu_0 \colon \lambda \in (0,1] \text{ and } \mu \in U\}$.

Claim 1. Assume that U is a lower semi-computable open subset of $\mathcal{P}(X)$, then the open set $\mathrm{ol}(U,\{\mu_0\})$ is also a lower semi-computable open set.

Since U is a lower semi-computable open set, then $\{U_\lambda \colon \lambda \in \mathbb{Q}\cap(0,1]\}$ is a sequence of uniformly lower semi-computable open sets, where $U_\lambda := \{\lambda u + (1-\lambda)\mu_0 \colon u \in U\}$. Hence, by Proposition 3.8, we obtain that $\bigcup_{\lambda\in\mathbb{Q}\cap(0,1]} U_\lambda$ is a lower semi-computable open set. On the other side, it follows from the definition of U_λ that $\bigcup_{\lambda\in\mathbb{Q}\cap(0,1]} U_\lambda = \mathrm{ol}(U,\{\mu_0\})$. Hence, $\mathrm{ol}(U,\{\mu_0\})$ is a lower semi-computable open set, establishing Claim 1.

By Proposition 3.18 (iii) and the recursive compactness of K, the number $\inf_{\mu\in K} W_\rho(\mu,\mu_0)$ is lower semi-computable. Hence, we can compute a rational number l which satisfies that $l < W_\rho(\mu,\mu_0)$ for each $\mu \in K$.

Claim 2. For each $m \in \mathbb{N}$ and each family $\mathcal{U} = \{\widetilde{B}_i : 1 \leqslant i \leqslant m \text{ and } i \in \mathbb{N}\}$ of ideal balls in $\mathcal{P}(X)$, $\mathcal{U}$ is a covering of K if and only if $\mathcal{U}' \cup \{B_{W_\rho}(\mu_0, l)\}$ is a covering of $\mathrm{cl}(K,\{\mu_0\})$, where $\mathcal{U}' = \{\mathrm{ol}(\widetilde{B}_i,\{\mu_0\}) : 1 \leqslant i \leqslant m \text{ and } i \in \mathbb{N}\}$.

We first assume that $K \subseteq \bigcup_{i=1}^m \widetilde{B}_i$ and show that $\mathrm{cl}(K,\{\mu_0\}) \subseteq \bigcup_{i=1}^m \mathrm{ol}(\widetilde{B}_i,\{\mu_0\}) \cup B_{W_\rho}(\mu_0, l)$. It is not hard to see that

$$\mathrm{cl}(K,\{\mu_0\}) = \{\mu_0\} \cup \mathrm{ol}(K,\{\mu_0\}) \subseteq B_{W_\rho}(\mu_0, l) \cup \left(\bigcup_{i=1}^m \mathrm{ol}(\widetilde{B}_i,\{\mu_0\})\right).$$

For the opposite direction, we assume that $\mathrm{cl}(K,\{\mu_0\}) \subseteq \bigcup_{i=1}^m \mathrm{ol}(\widetilde{B}_i,\{\mu_0\}) \cup B_{W_\rho}(\mu_0, l)$. Note that $l < W_\rho(\mu,\mu_0)$ for each $\mu \in K$. Then for each $\mu \in K$, there exists an integer $1 \leqslant i_\mu \leqslant m$ satisfying that $\mu \in \mathrm{ol}(\widetilde{B}_{i_\mu},\{\mu_0\})$, i.e., $\mu = \lambda\nu_\mu + (1-\lambda)\mu_0$ for some $\nu_\mu \in \widetilde{B}_{i_\mu}$ and $\lambda \in (0,1]$. By hypotheses in Proposition 5.8 (ii), we obtain that $\mu = \nu_\mu \in \widetilde{B}_{i_\mu}$ for each $\mu \in K$. Therefore, $K \subseteq \bigcup_{i=1}^m \widetilde{B}_i$. Claim 2 is now verified.

Let $\{B_i\}_{i\in\mathbb{N}}$ be the effective enumeration of ideal balls in $(\mathcal{P}(X),\, W_\rho,\, \mathcal{Q}_{\mathcal{S}})$. Assume that $\mathrm{cl}(K,\{\mu_0\})$ is recursively compact. Then by Definition 3.14, there exists an algorithm $M_0(\cdot)$ that satisfies the following property. On input a sequence $\{i_j\}_{j=1}^p$ of positive integers, the algorithm $M_0\bigl(\{i_j\}_{j=1}^p\bigr)$ halts if and only if $\mathrm{cl}(K,\{\mu_0\}) \subseteq \bigcup_{j=1}^p B_{i_j}$. In the following, to show that K is recursively compact, we establish an algorithm

$M_1(\cdot)$ such that on input a sequence $\{i_j\}_{j=1}^p$ of positive integers, the algorithm $M_1(\{i_j\}_{j=1}^p)$ halts if and only if $K \subseteq \bigcup_{j=1}^p B_{i_j}$.

Begin

(i) Read the sequences $\{i_j\}_{j=1}^p$.

(ii) By Claim 1, $\{\mathrm{ol}(B_{i_j}, \{\mu_0\})\}_{j=1}^p$ is sequence of uniformly lower semi-computable open sets. Since μ_0 is computable and $l \in \mathbb{Q}$, $B_{W_\rho}(\mu_0, l)$ is lower semi-computable open set. Hence, by Proposition 3.8, $V \coloneqq B_{W_\rho}(\mu_0, l) \cup \big(\bigcup_{j=1}^p \mathrm{ol}(B_{i_j}, \{\mu_0\})\big)$ is a lower semi-computable open set. Then we can compute a computable function $f\colon \mathbb{N} \to \mathbb{N}$ satisfying that $V = \bigcup_{n\in\mathbb{N}} B_{f(n)}$.

(iii) Set i to be 1.

(iv) **While** $i \geqslant 1$ **do**

  (1) Run the algorithm $M_0(\{f(j)\}_{j=1}^i)$.

  (2) Set v to be 1.

  (3) **While** $1 \leqslant v \leqslant i$ **do**

    (a) If $M_0(\{f(j)\}_{j=1}^v)$ halts, then set i to be -1.

    (b) Set v to be $v+1$.

  (4) Set i to be $i+1$.

End

To verify the feasibility of the algorithm, we fix a sequence $\{i_j\}_{j=1}^p$ and show that the algorithm $M_1(\{i_j\}_{j=1}^p)$ halts if and only if $K \subseteq \bigcup_{j=1}^p B_{i_j}$.

First, we assume that the algorithm $M_1(\{i_j\}_{j=1}^p)$ halts. Then by Step (iv) (3) (a), there exists an integer v such that $M_0(\{f(j)\}_{j=1}^v)$ halts. Thus, we obtain that $\mathrm{cl}(K, \{\mu_0\}) \subseteq \bigcup_{j=1}^v B_{f(j)} \subseteq V = B_{W_\rho}(\mu_0, l) \cup \big(\bigcup_{j=1}^p \mathrm{ol}(B_{i_j}, \{\mu_0\})\big)$ (see Step (ii)). Combining with Claim 2, this implies that $K \subseteq \bigcup_{j=1}^p B_{i_j}$.

For the opposite direction, we assume that $K \subseteq \bigcup_{j=1}^p$. Then by Claim 2, $\mathrm{cl}(K, \{\mu_0\}) \subseteq B_{W_\rho}(\mu_0, l) \cup \big(\bigcup_{j=1}^p \mathrm{ol}(B_{i_j}, \{\mu_0\})\big) = \bigcup_{n\in\mathbb{N}} B_{f(n)}$. Hence, there exists $v \in \mathbb{N}$ such that $\mathrm{cl}(K, \{\mu_0\}) \subseteq \bigcup_{j=1}^v B_{f(j)}$, i.e., the algorithm $M_0(\{f(j)\}_{j=1}^v)$ halts. Then by Step (iv) (3) (a), the algorithm $M_1(\{i_j\}_{j=1}^p)$ will halt eventually. ☐

As a corollary of Proposition 5.8, we establish the following conclusion.

Corollary 5.9. *Let $(X, \rho, \mathcal{S})$ be a computable metric space and X be a recursively compact set. Assume that $\{\mu_i\}_{i=1}^N$ is a sequence of uniformly computable points in $\mathcal{P}(X)$ for some $N \in \mathbb{N}$. Define $K_0 \coloneqq K$ and $K_i = \mathrm{cl}(K_{i-1}, \{\mu_i\})$ for each integer $1 \leqslant i \leqslant N$, recursively. Then the following statements hold:*

(i) *If K is recursively compact, then K_N is also recursively compact.*

(ii) *Assume that the set K and the sequence $\{\mu_i\}_{i=1}^N$ satisfy the following properties:*

  (a) *$K_i \subseteq \mathcal{P}(X)$ is compact for each integer $1 \leqslant i \leqslant N$,*

  (b) *K_N is a recursively compact set in $(\mathcal{P}(X), W_\rho, \mathcal{Q}_\mathcal{S})$,*

  (c) *for each $y \in \mathcal{P}(X)$ and each sequence $\{r_i\}_{i=0}^N$ of positive numbers with $\sum_{i=0}^N r_i = 1$, if $r_0 y + \sum_{i=1}^N r_i \mu_i \in K_N$, then $r_0 = 1$,*

  then K is a recursively compact set.

Proof. (i) Since $K_0 = K$ is recursively compact and μ_1 is a computable point, by Proposition 5.8 (i), $K_1 = \mathrm{cl}(K_0, \{\mu_1\})$ is recursively compact. Hence, by Proposition 5.8 (i) and the uniform computability of $\{\mu_i\}_{i=1}^N$, we show that K_i is recursively compact for each integer $1 \leqslant i \leqslant N$ by induction.

(ii) First, we consider $y \in \mathcal{P}(X)$ and $\lambda \in [0,1]$ with $\lambda y + (1-\lambda)\mu_N$. Indeed, it follows from statement (ii) (c) that $\lambda = 1$. Combining with the facts that K_{N-1} is compact, $K_N = \mathrm{cl}(K_{N-1}, \{\mu_N\})$ is a recursively compact, and μ_N is a computable point, by Proposition 5.8 (ii), this implies that K_{N-1} is recursively compact. Hence, by Proposition 5.8 (ii), the uniform computability of $\{\mu_i\}_{i=1}^N$, we show that

K_{N-i} is recursively compact for each integer $1 \leqslant i \leqslant N$ by induction. Therefore, $K = K_0$ is recursively compact. $\square$

The following result generalizes [Zie06, Theorem 18 (d)]. The proof is essentially the same, and we include it for the convenience of the reader.

Lemma 5.10. *Let $(X, \rho, \mathcal{S})$ be a computable metric space with $\mathcal{S} = \{s_i\}_{i\in\mathbb{N}}$, $\{q_j\}_{j\in\mathbb{N}}$ be an effective enumeration of $\mathbb{Q}^+$, and $T\colon X \to X$ be a computable map. Assume that $\{U_i\}_{i\in\mathbb{N}}$ is a sequence of uniformly lower semi-computable open sets of X on which T is injective and open, and $\{O_j\}_{j\in\mathbb{N}}$ is a sequence of uniformly lower semi-computable open sets such that for each $j \in \mathbb{N}$, $O_j \subseteq U_i$ for some $i \in \mathbb{N}$. Then $\{T(O_j)\}_{j\in\mathbb{N}}$ is a sequence of uniformly lower semi-computable open sets.*

Proof. Given $j \in \mathbb{N}$, we compute a function $u_j\colon \mathbb{N} \to \mathbb{N}$ satisfying that $\{s_{u_j(m)}\}_{m\in\mathbb{N}}$ is the set of all ideal points contained in O_j. Note that by Definition 3.3, $\mathcal{S}$ is dense in X. Then the set $\{s_{u_j(m)}\}_{m\in\mathbb{N}}$ is dense in O_j for each $j \in \mathbb{N}$. Since T is computable, $\{T(s_{u_j(m)})\}_{j,\,m\in\mathbb{N}}$ is a sequence of uniformly computable points. Moreover, since T is continuous, injective, and open on U_i for each $i \in \mathbb{N}$, $T|_{U_i}\colon U_i \to T(U_i)$ is a homeomorphism. Then for each $j \in \mathbb{N}$ with $O_j \subseteq U_i$ for some $i \in \mathbb{N}$, by the fact that $\{s_{u_j(m)}\}_{m\in\mathbb{N}}$ is dense in O_j, we have

$$T(O_j) = \bigcup_{m\in\mathbb{N}} B_\rho\bigl(T\bigl(s_{u_j(m)}\bigr), \rho\bigl(T\bigl(s_{u_j(m)}\bigr), T(\partial O_j)\bigr)\bigr). \tag{5.21}$$

By Definition 3.14, $\{\partial O_j\}_{j\in\mathbb{N}}$ is a sequence of recursively compact sets. Combining with the fact that T is computable and [GHR11, Proposition 2.6.3], this implies that $\{T(\partial O_j)\}_{j\in\mathbb{N}}$ is a sequence of recursively compact sets. Hence, by Proposition 3.18 (iii), $\{\rho\bigl(T\bigl(s_{u_j(m)}\bigr), T(\partial O_j)\bigr)\}_{j,\,m\in\mathbb{N}}$ is a sequence of uniformly lower semi-computable real numbers. Then by Remark 3.7), Definitions 3.2, and 3.6, $\{B_\rho\bigl(T\bigl(s_{u_j(m)}\bigr), \rho\bigl(T\bigl(s_{u_j(m)}\bigr), T(\partial O_j)\bigr)\bigr)\}_{j,\,m\in\mathbb{N}}$ is a sequence of uniformly lower semi-computable open sets. By constructing a computable bijection between $\mathbb{N}^2$ and $\mathbb{N}$, it is not hard to derive from (5.21) that $\{T(O_j)\}_{j\in\mathbb{N}}$ is a sequence of uniformly lower semi-computable open sets. This completes the proof of Lemma 5.10. $\square$

With these preparations, we now prove Theorem 1.3.

Proof of Theorem 1.3. Let $\{\varphi_m\}_{m\in\mathbb{N}}$ be the effective enumeration of $\mathfrak{E}(\mathcal{S})$ (recall Remark 3.25). Since $\{U_j\}_{j\in\mathbb{N}}$ is a sequence of uniformly lower semi-computable open sets, by Proposition 3.23, there exists a sequence $h_{j,k}\colon X \to \mathbb{R}$, j, $k \in \mathbb{N}$, of uniformly computable functions satisfying that for each $j \in \mathbb{N}$, the following two properties hold:

(i) $\{h_{j,k}(x)\}_{k\in\mathbb{N}}$ is non-decreasing and $h_{j,k}(x) \to \mathbb{1}_{U_j}(x)$ as $k \to +\infty$ for each $x \in X$,

(ii) $h_{j,k}(x) = 0$ for each $x \notin U_j$ and each $k \in \mathbb{N}$.

Since $J\colon X \to \mathbb{R}$ is upper semi-computable on X, by Definition 3.12, there exists a sequence $J_l\colon X \to \mathbb{R}$, $l \in \mathbb{N}$, of uniformly computable functions on X satisfying that for each $x \in X$, $\{J_l(x)\}_{l\in\mathbb{N}}$ is non-increasing and $J_l(x) \to J(x)$ as $l \to +\infty$. Note that $h_{j,k}(x) = 0$ for each pair of j, $k \in \mathbb{N}$, and each point $x \notin U_j$. Since for each j, k, l, $m \in \mathbb{N}$, T is injective on U_j (see property (b) in Theorem 1.3), we can define

$$y_{j,k,m}(x) := \begin{cases} \bigl((\varphi_m \cdot h_{j,k}) \circ (T|_{U_j})^{-1}\bigr)(x) & \text{if } x \in T(U_j); \\ 0 & \text{if } x \notin T(U_j) \end{cases} \quad \text{and} \tag{5.22}$$

$$\Psi^{j,k}_{l,m} := \biggl\{\mu \in \mathcal{P}(X) : \int y_{j,k,m}\,\mathrm{d}\mu > \int (\varphi_m \cdot J_l \cdot h_{j,k})\,\mathrm{d}\mu\biggr\}. \tag{5.23}$$

We first establish the following claim.

Claim 1. $\bigl\{\Psi^{j,k}_{l,m} : j,\, k,\, l,\, m \in \mathbb{N}\bigr\}$ is a sequence of uniformly lower semi-computable open sets.

By Corollary 3.21, it follows from the uniform computability of $\{\varphi_m\}_{m\in\mathbb{N}}$, $\{J_l\}_{l\in\mathbb{N}}$, and $\{h_{j,k} : j,\, k \in \mathbb{N}\}$ that $\big\{\mathcal{V}^{j,k}_{l,m} : j,\, k,\, l,\, m \in \mathbb{N}\big\}$ is a sequence of uniformly computable functions on $\mathcal{P}(X)$, where for each $j,\, k,\, l,\, m \in \mathbb{N}$,

$$\mathcal{V}^{j,k}_{l,m}(\mu) \coloneqq \int (\varphi_m \cdot J_l \cdot h_{j,k}) \,\mathrm{d}\mu \quad \text{for each } \mu \in \mathcal{P}(X).$$

Let $\{B_v\}_{v\in\mathbb{N}}$ be the effective enumeration of ideal balls in the metric space $(\mathbb{R},\, d_{\mathrm{E}},\, \mathbb{Q})$, where d_{E} denotes the Euclidean metric on $\mathbb{R}$. By Proposition 3.11, to establish that $\big\{y_{j,k,m} : j,\, k,\, m \in \mathbb{N}\big\}$ is a sequence of uniformly computable functions on X, it suffices to show that $\{y^{-1}_{j,k,m}(B_v) : j,\, k,\, m,\, v \in \mathbb{N}\}$ is a sequence of lower semi-computable open sets. Since $\{B_v\}_{v\in\mathbb{N}}$ is a sequence of ideal balls, then there exists an algorithm $\mathcal{A}(\cdot)$ such that for each $v \in \mathbb{N}$, on input $v \in \mathbb{N}$, $\mathcal{A}(v)$ outputs 0 if $0 \in B_v$, and outputs 1 if $0 \notin B_v$. For each $v \in \mathbb{N}$, depending on the output of the algorithm $\mathcal{A}(v)$, we have the following two cases.

Case 1: The algorithm $\mathcal{A}(v)$ outputs 1.

In this case, we have $0 \notin B_v$. Hence, by (5.22), we obtain that $y^{-1}_{j,k,m}(B_v) = T((\varphi_m \cdot h_{j,k})^{-1}(B_v))$. Note that by $0 \notin B_v$, it follows from the uniform computability of $\{\varphi_m\}_{m\in\mathbb{N}}$ and $\{h_{j,k} : j,\, k \in \mathbb{N}\}$ that $\{(\varphi_m \cdot h_{j,k})^{-1}(B_v) : j,\, k,\, m,\, v \in \mathbb{N}\}$ is a sequence of uniformly lower semi-computable open sets contained in U_j. By Lemma 5.10, it follows from the hypotheses and the computability of T that $y^{-1}_{j,k,m}(B_v) = T((\varphi_m \cdot h_{j,k})^{-1}(B_v))$ is a lower semi-computable open set.

Case 2: The algorithm $\mathcal{A}(v)$ outputs 0.

In this case, we have $0 \in B_v$. Hence, by (5.22), we obtain that

$$y^{-1}_{j,k,m}(B_v) = T(U_j)^c \cup T(U_j \cap (\varphi_m \cdot h_{j,k})^{-1}(B_v)) = (T(((\varphi_m \cdot h_{j,k})^{-1}(B_v))^c))^c.$$

Thus, by Proposition 3.18 (ii) and [GHR11, Proposition 2.6.3], $y^{-1}_{j,k,m}(B_v)$ is a lower semi-computable open set. Therefore, $\big\{y^{-1}_{j,k,m}(B_v) : j,\, k,\, m,\, v \in \mathbb{N}\big\}$ is a sequence of lower semi-computable open sets.

Hence, by Corollary 3.21, $\mathcal{W}^{j,k}_m \colon \mathcal{P}(X) \to \mathbb{R}$, $j,\, k,\, m \in \mathbb{N}$, is a sequence of uniformly computable functions on $\mathcal{P}(X)$, where

$$\mathcal{W}^{j,k}_m(\mu) \coloneqq \int y_{j,k,m} \,\mathrm{d}\mu \quad \text{for each } \mu \in \mathcal{P}(X) \text{ and each } j,\, k,\, m \in \mathbb{N}.$$

By (5.23), we obtain that $\Psi^{j,k}_{l,m} = \mathcal{P}(X) \smallsetminus \big(\mathcal{W}^{j,k}_m - \mathcal{V}^{j,k}_{l,m}\big)^{-1}(\mathbb{R}^+)$. Then by Proposition 3.11, $\big\{\Psi^{j,k}_{l,m} : j,\, k,\, l,\, m \in \mathbb{N}\big\}$ is a sequence of uniformly lower semi-computable open sets. So Claim 1 follows.

Set $\Psi \coloneqq (\mathcal{M}(X,T) \cap \mathcal{P}(X,C)) \smallsetminus \big(\bigcup_{j,\,k,\,l,\,m\in\mathbb{N}} \Psi^{j,k}_{l,m}\big)$. We now prove the following claim.

Claim 2. For each $\mu \in \mathcal{M}(X,T)$, $\mu \in \Psi$ if and only if

$$\mu(T(A)) \leqslant \int_A J \,\mathrm{d}\mu \tag{5.24}$$

for each Borel set $A \subseteq C \smallsetminus \operatorname{Sing}(T)$ satisfying that $T(A)$ is Borel and T is injective on A.

We first show the forward implication of Claim 2. Consider a measure $\mu \in \Psi$. Then by (5.22) and (5.23),

$$\int_{T(U_j)} \big((\varphi_m \cdot h_{j,k}) \circ (T|_{U_j})^{-1}\big) \,\mathrm{d}\mu \leqslant \int (\varphi_m \cdot J_l \cdot h_{j,k}) \,\mathrm{d}\mu \quad \text{for all } j,\, k,\, l,\, m \in \mathbb{N}. \tag{5.25}$$

By letting integers k, l tend to $+\infty$ in (5.25) and applying Lebesgue Convergence Theorem, it follows from the properties (i) and (ii) of $\{h_{j,k} : j,\, k \in \mathbb{N}\}$ that

$$\int_{T(U_j)} \big(\varphi_m \circ (T|_{U_j})^{-1}\big) \,\mathrm{d}\mu \leqslant \int_{U_j} (\varphi_m \cdot J) \,\mathrm{d}\mu \quad \text{for all } j,\, m \in \mathbb{N}. \tag{5.26}$$

Note that for each $j \in \mathbb{N}$, $\mu_{1,j}$ and $\mu_{2,j}$ are both Borel finite measures on X, where $\mu_{1,j}$ and $\mu_{2,j}$ are defined by

$$\mu_{1,j}(A) \coloneqq \mu(T(A \cap U_j)) \quad \text{and} \quad \mu_{2,j}(A) \coloneqq \int_{A\cap U_j} J \,\mathrm{d}\mu$$

for each Borel subset $A \subseteq X$. By the change of variable formula, it is not hard to derive from (5.25) that $\langle \mu_{1,j}, \varphi_m \rangle \leqslant \langle \mu_{2,j}, \varphi_m \rangle$ for each j, $m \in \mathbb{N}$. Hence, by Proposition 3.26, it follows from (5.26) that for each Borel subset $A \subseteq X$ and each $j \in \mathbb{N}$,

$$\mu(T(A \cap U_j)) \leqslant \int_{A \cap U_j} J \, \mathrm{d}\mu.$$

Note that by property (a) in Theorem 1.3, $C \smallsetminus \mathrm{Sing}(T) = \bigcup_{j \in \mathbb{N}} U_j$. Then we obtain $\mu(T(A)) \leqslant \int_A J \, \mathrm{d}\mu$ whenever $A \subseteq C \smallsetminus \mathrm{Sing}(T)$ is a Borel subset, for which $T(A)$ is Borel and T is injective on A.

We now prove the backward implication of Claim 2. Assume that the measure $\mu \in M(X,T)$ satisfies that $\mu(T(A)) \leqslant \int_A J \, \mathrm{d}\mu$ for each Borel set $A \subseteq C \smallsetminus \mathrm{Sing}(T)$ satisfying that $T(A)$ is Borel and T is injective on A. By (5.23), it suffices to prove that $\mu \notin \Psi_{l,m}^{j,k}$ for all j, k, l, $m \in \mathbb{N}$. Note that T is injective on U_j for each $j \in \mathbb{N}$, and $J(x) \leqslant J_l(x)$ for each $x \in X$ and each $l \in \mathbb{N}$. Then by (5.22), we obtain that for all j, k, l, $m \in \mathbb{N}$,

$$\begin{aligned} \int y_{j,\,k,\,m} \, \mathrm{d}\mu &= \int_{T(U_j)} \big((\varphi_m \cdot h_{j,k}) \circ (T|_{U_j})^{-1} \big) \, \mathrm{d}\mu = \int_{U_j} (\varphi_m \cdot h_{j,k}) \, \mathrm{d}(\mu \circ (T|_{U_j})) \\ &\leqslant \int_{U_j} (\varphi_m \cdot J \cdot h_{j,k}) \, \mathrm{d}\mu \leqslant \int (\varphi_m \cdot J_l \cdot h_{j,k}) \, \mathrm{d}\mu. \end{aligned}$$

By (5.23), we have $\mu \notin \Psi_{l,m}^{j,k}$ for all j, k, l, $m \in \mathbb{N}$. Therefore, we complete the proof of Claim 2.

Claim 3. $\mathcal{E}(T, \phi) \cap \mathcal{P}(X, C \smallsetminus \mathrm{Sing}(T)) = \Psi \cap \mathcal{P}(X, C \smallsetminus \mathrm{Sing}(T))$.

Denote by $\mathcal{M}(X, T, J, C \smallsetminus \mathrm{Sing}(T))$ the set of all probability measures $\mu \in \mathcal{M}(X,T)$ satisfying that the function J is a Jacobian on $C \smallsetminus \mathrm{Sing}(T)$ for T with respect to μ. By Corollary 5.6, we obtain that $\mathcal{E}(T, \phi) \cap \mathcal{P}(X, C \smallsetminus \mathrm{Sing}(T)) = \mathcal{M}(X, T, J, C \smallsetminus \mathrm{Sing}(T))$. Thus, to establish Claim 3, it suffices to show that $\mathcal{M}(X, T, J, C \smallsetminus \mathrm{Sing}(T)) = \Psi \cap \mathcal{P}(X, C \smallsetminus \mathrm{Sing}(T))$.

Fix an arbitrary measure $\mu \in \Psi \cap \mathcal{P}(X, C \smallsetminus \mathrm{Sing}(T))$. In particular, $\mu(C \smallsetminus \mathrm{Sing}(T)) = 1$. By Proposition 5.3, there exists a μ-measurable function $J_\mu \colon X \to [0, +\infty)$ and a Borel set E with full μ-measure satisfying that J_μ is a Jacobian on E for T with respect to μ. By Claim 2, it follows from $\mu \in \mathcal{M}(X,T)$ that for each μ-measurable set $A \subseteq C \smallsetminus \mathrm{Sing}(T)$ satisfying that $T(A)$ is μ-measurable and T is injective on A, $\mu(A) = 0$ if and only if $\mu(T(A)) = 0$. Combining with (5.1), this implies that

$$\mu(T(A)) = \mu(T(A \cap E)) + \mu(T(A \cap E^c)) = \int_{A \cap E} J_\mu \, \mathrm{d}\mu + 0 = \int_A J_\mu \, \mathrm{d}\mu.$$

Thus, J_μ is a Jacobian on $C \smallsetminus \mathrm{Sing}(T)$ for T with respect to μ. Moreover, by Claim 2, $J(x) \geqslant J_\mu(x)$ for μ-a.e. $x \in X$. By Theorem 5.5 and property (i) in Theorem 1.3, we have that

$$\sum_{y \in T^{-1}(x) \cap (C \smallsetminus \mathrm{Sing}(T))} \frac{1}{J(y)} = 1 = \sum_{y \in T^{-1}(x) \cap (C \smallsetminus \mathrm{Sing}(T))} \frac{1}{J_\mu(y)}$$

for μ-a.e. $x \in X$. Then it follows from the T-invariance of μ that $J(x) = J_\mu(x)$ for μ-a.e. $x \in X$, i.e., $\mu \in \mathcal{M}(X, T, J, C \smallsetminus \mathrm{Sing}(T))$. Thus, we obtain that $\Psi \cap \mathcal{P}(X, C \smallsetminus \mathrm{Sing}(T)) \subseteq \mathcal{M}(X, T, J, C \smallsetminus \mathrm{Sing}(T))$.

For the opposite direction, it follows from Claim 2 that $\mathcal{M}(X, T, J, C \smallsetminus \mathrm{Sing}(T)) \subseteq \Psi \cap \mathcal{P}(X, C \smallsetminus \mathrm{Sing}(T))$. Therefore, Claim 3 is now verified.

By [LS24, Lemma 6.3 (iii)], the set of T-invariant Borel probability measures supported on a periodic orbit of T containing singular points for T can be written as

$$\left\{ \frac{1}{n} \sum_{i=1}^{n} \delta_{T^i(x)} : x \text{ is a singular periodic point for } T \text{ with period } n \in \mathbb{N} \right\}. \tag{5.27}$$

Since $\mathrm{card}(\mathrm{Sing}(T)) < +\infty$, the set defined by (5.27) is finite, say $\big\{ \mu_i = \frac{1}{n_i} \sum_{k=1}^{n_i} \delta_{T^k(x_i)} \big\}_{i=1}^{N}$, where x_i is a singular periodic point for T with period $n_i \in \mathbb{N}$. Denote by $\mathcal{M}^*$ the set

$$\mathcal{M}^* := \left\{ \sum_{i=1}^{N} r_i \mu_i : \sum_{i=1}^{N} r_i = 1, \text{ where } r_i \in [0,1] \text{ for each integer } 1 \leqslant i \leqslant N \right\}$$

and by $\mathrm{Orb}_{\mathrm{sp}}$ the union of all periodic orbits containing singular points. Then $\mathrm{card}(\mathrm{Orb}_{\mathrm{sp}}) < +\infty$. By considering the ergodic decomposition, it is not hard to derive from [LS24, Lemma 6.3 (iii)] that $\mathcal{M}(X,T) \cap \mathcal{P}(X, \mathrm{Orb}_{\mathrm{sp}}) = \mathcal{M}^*$.

Now we fix a sequence $\{r_i\}_{i=1}^N$ of non-negative numbers with $\sum_{i=1}^N r_i = 1$ and $\mu_0 = \sum_{i=1}^N r_i\mu_i \in \Psi$. We consider an integer i with $r_i > 0$ and show that $\mu_i \in \Psi$. Recall that $\mu_i = \frac{1}{n_i}\sum_{k=1}^{n_i} \delta_{T^k(x_i)}$. To show that $\mu_i \in \Psi$, we consider a point $y \in C \smallsetminus \mathrm{Sing}(T)$ that satisfies that $T(y) = T^k(x_i)$ for some $k \in \mathbb{N}$. Since different periodic orbits are disjoint, we have $\mu_j(\{T(y)\}) = \mu_j(\{y\}) = 0$ for each integer $j \in [1,N] \smallsetminus \{i\}$. Combining with $\mu_0 \in \Psi$, by Claim 2, this implies that

$$\mu_i(\{T(y)\}) = \frac{\mu_0(\{T(y)\})}{r_i} \leqslant \frac{1}{r_i}\int_{\{y\}} J \,\mathrm{d}\mu_0 = \int_{\{y\}} J \,\mathrm{d}\mu_i.$$

Then by Claim 2, we have $\mu_i \in \Psi$. Hence, we have

$$\mathcal{M}^* \cap \Psi = \left\{ \sum_{k=1}^M l_k \mu_{i_k} : \sum_{k=1}^M l_k = 1, \text{ where } l_k \in [0,1] \text{ for each integer } 1 \leqslant k \leqslant M \right\}, \tag{5.28}$$

where $\{\mu_{i_k} : k \in \mathbb{N} \cap [1,M]\} = \{\mu_i : i \in \mathbb{N}\cap[1,N]\} \cap \Psi$.

Claim 4. $\Psi = \mathrm{cl}(\mathcal{M}^* \cap \Psi, \mathcal{P}(X, C \smallsetminus \mathrm{Sing}(T)) \cap \Psi)$.

We first establish that $\Psi \subseteq \mathrm{cl}(\mathcal{M}^* \cap \Psi, \mathcal{P}(X, C \smallsetminus \mathrm{Sing}(T)) \cap \Psi)$. We consider a measure $\mu \in \Psi$ and show that $\mu \in \mathrm{cl}(\mathcal{M}^* \cap \Psi, \mathcal{P}(X, C \smallsetminus \mathrm{Sing}(T)) \cap \Psi)$. Depending on $\lambda_\mu := \mu(\mathrm{Orb}_{\mathrm{sp}})$, we have the following three cases.

Case 1: $\lambda_\mu \neq 0$ and $\lambda_\mu \neq 1$.

In this case, we define two measures μ_a and μ_b by

$$\mu_a(A) = \frac{\mu(A \cap \mathrm{Orb}_{\mathrm{sp}})}{\lambda_\mu} \quad \text{and} \quad \mu_b(A) = \frac{\mu(A \smallsetminus \mathrm{Orb}_{\mathrm{sp}})}{1-\lambda_\mu} \tag{5.29}$$

for each μ-measurable set $A \subseteq X$, respectively. Then μ_a and μ_b are both Borel probability measures such that $\mu_a(\mathrm{Orb}_{\mathrm{sp}}) = \mu_b((\mathrm{Orb}_{\mathrm{sp}})^c) = 1$ and $\mu = \lambda_\mu\mu_a + (1-\lambda_\mu)\mu_b$. Now we show that μ_a is T-invariant. To this end, we fix a μ-measurable set $A \subseteq X$ and show that $\mu_a(T^{-1}(A)) = \mu_a(A)$. Since $T(\mathrm{Orb}_{\mathrm{sp}}) = \mathrm{Orb}_{\mathrm{sp}}$, $T^{-1}(A) \cap \mathrm{Orb}_{\mathrm{sp}} \subseteq T^{-1}(A \cap \mathrm{Orb}_{\mathrm{sp}})$. Hence, $\mu\bigl(T^{-1}(A)\cap\mathrm{Orb}_{\mathrm{sp}}\bigr) \leqslant \mu\bigl(T^{-1}(A\cap\mathrm{Orb}_{\mathrm{sp}})\bigr)$. Since μ is T-invariant, $\mu\bigl(T^{-1}(\mathrm{Orb}_{\mathrm{sp}}) \smallsetminus \mathrm{Orb}_{\mathrm{sp}}\bigr) = \mu\bigl(T^{-1}(\mathrm{Orb}_{\mathrm{sp}})\bigr) - \mu(\mathrm{Orb}_{\mathrm{sp}}) = 0$. Note that $T^{-1}(A\cap\mathrm{Orb}_{\mathrm{sp}}) \smallsetminus \bigl(T^{-1}(A)\cap\mathrm{Orb}_{\mathrm{sp}}\bigr) \subseteq T^{-1}(\mathrm{Orb}_{\mathrm{sp}}) \smallsetminus \mathrm{Orb}_{\mathrm{sp}}$. Then we obtain that

$$\begin{aligned} 0 &\leqslant \mu\bigl(T^{-1}(A \cap \mathrm{Orb}_{\mathrm{sp}})\bigr) - \mu\bigl(T^{-1}(A) \cap \mathrm{Orb}_{\mathrm{sp}}\bigr) = \mu\bigl(T^{-1}(A \cap \mathrm{Orb}_{\mathrm{sp}}) \smallsetminus \bigl(T^{-1}(A) \cap \mathrm{Orb}_{\mathrm{sp}}\bigr)\bigr) \\ &\leqslant \mu\bigl(T^{-1}(\mathrm{Orb}_{\mathrm{sp}}) \smallsetminus \mathrm{Orb}_{\mathrm{sp}}\bigr) = 0, \end{aligned}$$

which implies that $\mu\bigl(T^{-1}(A\cap\mathrm{Orb}_{\mathrm{sp}})\bigr) = \mu\bigl(T^{-1}(A)\cap\mathrm{Orb}_{\mathrm{sp}}\bigr)$. Combining the T-invariance of μ and (5.29), this implies that

$$\mu_a\bigl(T^{-1}(A)\bigr) = \frac{\mu(T^{-1}(A) \cap \mathrm{Orb}_{\mathrm{sp}})}{\lambda_\mu} = \frac{\mu(T^{-1}(A \cap \mathrm{Orb}_{\mathrm{sp}}))}{\lambda_\mu} = \frac{\mu(A \cap \mathrm{Orb}_{\mathrm{sp}})}{\lambda_\mu} = \mu_a(A).$$

Thus, $\mu_a \in \mathcal{M}(X,T)$. Combining with $\mu = \lambda_\mu\mu_a + (1-\lambda_\mu)\mu_b$ and $\mu \in \mathcal{M}(X,T)$, this implies that $\mu_b \in \mathcal{M}(X,T)$.

Next, we show that μ_a, $\mu_b \in \Psi$. To this end, by Claim 2, we fix a μ-measurable set $A \subseteq C \smallsetminus \mathrm{Sing}(T)$ satisfying that $T(A)$ is μ-measurable and T is injective on A and show that $\mu_a(T(A)) \leqslant \int_A J \,\mathrm{d}\mu_a$ and $\mu_b(T(A)) \leqslant \int_A J \,\mathrm{d}\mu_b$. Since $T(\mathrm{Orb}_{\mathrm{sp}}) = \mathrm{Orb}_{\mathrm{sp}}$, $T(A \cap \mathrm{Orb}_{\mathrm{sp}}) \subseteq T(A) \cap \mathrm{Orb}_{\mathrm{sp}}$. Hence, $\mu(T(A\cap\mathrm{Orb}_{\mathrm{sp}})) \leqslant \mu(T(A)\cap\mathrm{Orb}_{\mathrm{sp}})$. Since μ is T-invariant, $0 \leqslant \mu\bigl(\bigl(A\cap T^{-1}(\mathrm{Orb}_{\mathrm{sp}})\bigr) \smallsetminus \mathrm{Orb}_{\mathrm{sp}}\bigr) \leqslant \mu\bigl(T^{-1}(\mathrm{Orb}_{\mathrm{sp}}) \smallsetminus \mathrm{Orb}_{\mathrm{sp}}\bigr) = \mu\bigl(T^{-1}(\mathrm{Orb}_{\mathrm{sp}})\bigr) - \mu(\mathrm{Orb}_{\mathrm{sp}}) = 0$. Hence, we have $\mu\bigl((A\cap T^{-1}(\mathrm{Orb}_{\mathrm{sp}})\bigr) = 0$. Combining with Claim 2 and the fact that $\mu \in \Psi$, this implies that

$$0 \leqslant \mu(T(A \smallsetminus \mathrm{Orb}_{\mathrm{sp}}) \cap \mathrm{Orb}_{\mathrm{sp}}) \leqslant \int_{(A\cap T^{-1}(\mathrm{Orb}_{\mathrm{sp}}))\smallsetminus\mathrm{Orb}_{\mathrm{sp}}} J \,\mathrm{d}\mu = 0. \tag{5.30}$$

Since T is injective on A, $T(A \smallsetminus \mathrm{Orb}_{\mathrm{sp}}) \cap \mathrm{Orb}_{\mathrm{sp}} = (T(A) \cap \mathrm{Orb}_{\mathrm{sp}}) \smallsetminus (T(A \cap \mathrm{Orb}_{\mathrm{sp}}))$. Combining with (5.30), this implies that $\mu(T(A) \cap \mathrm{Orb}_{\mathrm{sp}}) - \mu(T(A \cap \mathrm{Orb}_{\mathrm{sp}})) = \mu(T(A \smallsetminus \mathrm{Orb}_{\mathrm{sp}}) \cap \mathrm{Orb}_{\mathrm{sp}}) = 0$. Hence, it follows from (5.29), Claim 2, and the fact that $\mu \in \Psi$ that

$$\mu_a(T(A)) = \frac{\mu(T(A) \cap \mathrm{Orb}_{\mathrm{sp}})}{\lambda_\mu} = \frac{\mu(T(A \cap \mathrm{Orb}_{\mathrm{sp}}))}{\lambda_\mu} \leqslant \frac{\int_{A \cap \mathrm{Orb}_{\mathrm{sp}}} J \,\mathrm{d}\mu}{\lambda_\mu} = \int_A J \,\mathrm{d}\mu_a. \tag{5.31}$$

Moreover, since T is injective on A, $\mu(T(A \smallsetminus \mathrm{Orb}_{\mathrm{sp}})) = \mu(T(A)) - \mu(T(A) \cap \mathrm{Orb}_{\mathrm{sp}}) = \mu(T(A)) - \mu(T(A \cap \mathrm{Orb}_{\mathrm{sp}})) = \mu(T(A \smallsetminus \mathrm{Orb}_{\mathrm{sp}}))$. Hence, it follows from (5.29), Claim 2, and the fact that $\mu \in \Psi$ that

$$\mu_b(T(A)) = \frac{\mu(T(A) \smallsetminus \mathrm{Orb}_{\mathrm{sp}})}{1-\lambda_\mu} = \frac{\mu(T(A \smallsetminus \mathrm{Orb}_{\mathrm{sp}}))}{1-\lambda_\mu} \leqslant \frac{\int_{A \smallsetminus \mathrm{Orb}_{\mathrm{sp}}} J \,\mathrm{d}\mu}{1-\lambda_\mu} = \int_A J \,\mathrm{d}\mu_b. \tag{5.32}$$

By (5.31) and (5.32), it follows from Claim 2 that μ_a, $\mu_b \in \Psi$.

Case 2: $\lambda_\mu = 0$.

In this case, we have $\mu(\mathrm{Orb}_{\mathrm{sp}}) = 0$. Hence, $\mu \in \Psi \cap \mathcal{P}(X, C \smallsetminus \mathrm{Sing}(T)) \subseteq \mathrm{cl}(\mathcal{M}^* \cap \Psi, \mathcal{P}(X, C \smallsetminus \mathrm{Sing}(T)) \cap \Psi)$.

Case 3: $\lambda_\mu = 1$.

In this case, we have $\mu(\mathrm{Orb}_{\mathrm{sp}}) = 1$. Hence, $\mu \in \Psi \cap \mathcal{P}(X, \mathrm{Orb}_{\mathrm{sp}}) = \mathcal{M}^* \cap \Psi \subseteq \mathrm{cl}(\mathcal{M}^* \cap \Psi, \mathcal{P}(X, C \smallsetminus \mathrm{Sing}(T)) \cap \Psi)$.

Therefore, we show that $\Psi \subseteq \mathrm{cl}(\mathcal{M}^* \cap \Psi, \mathcal{P}(X, C \smallsetminus \mathrm{Sing}(T)) \cap \Psi)$.

Next, we show that $\mathrm{cl}(\mathcal{M}^* \cap \Psi, \mathcal{P}(X, C \smallsetminus \mathrm{Sing}(T)) \cap \Psi) \subseteq \Psi$. We now consider a positive number $\lambda \in [0,1]$, measures $\nu_a \in \mathcal{M}^* \cap \Psi$, and $\nu_b \in \Psi \cap \mathcal{P}(X, C \smallsetminus \mathrm{Sing}(T))$. By Claim 2, it follows from ν_a, $\nu_b \in \Psi$ that $\lambda\nu_a + (1-\lambda)\nu_b \in \Psi$. Therefore, we complete the proof of Claim 4.

Now we turn to prove Theorem 1.3. By Claims 3 and 4, we obtain that

$$\Psi = \mathrm{cl}(\mathcal{M}^* \cap \Psi, \mathcal{E}(T,\phi) \cap \mathcal{P}(X, C \smallsetminus \mathrm{Sing}(T))). \tag{5.33}$$

By [BHLZ24, Lemma 4.12], it follows from the computability of the map $T\colon X \to X$ that $\mathcal{M}(X,T)$ is a recursively compact set. By Proposition 3.8 and Claim 1, $\bigcup_{j,k,l,m \in \mathbb{N}} \Psi^{j,k}_{l,m}$ is a lower semi-computable open set. Moreover, by Proposition 3.20, it follows from the recursive compactness of C that $\mathcal{P}(X,C)$ is a recursively compact set. Hence, by Proposition 3.18 (ii) and the definition of Ψ, we obtain that Ψ is a recursively compact set. Hence, by (5.33) and (5.28), we apply Corollary 5.9 (ii) to the recursively compact set Ψ and the sequence $\{\mu_{i_k}\}_{k=1}^M$. Then we obtain that $\mathcal{E}(T,\phi) \cap \mathcal{P}(X, C \smallsetminus \mathrm{Sing}(T))$ is recursively compact.

Now we consider a sequence $\{g_i\}_{i=1}^N$ of non-negative numbers with $\sum_{i=1}^N g_i = 1$ and $\sum_{i=1}^N g_i\mu_i \in \mathcal{E}(T,\phi) \cap \mathcal{P}(X,C) \cap \mathcal{M}^*$. Then by considering the ergodic decomposition, it follows from Claim 2 that for each integer $1 \leqslant i \leqslant N$, if $g_i \neq 0$, then $\mu_i \in \mathcal{E}(T,\phi) \cap \mathcal{P}(X,C) \cap \mathcal{M}^*$. Hence, we have

$$\mathcal{E}(T,\phi) \cap \mathcal{P}(X,C) \cap \mathcal{M}^* = \left\{ \sum_{k=1}^{M'} l_k \mu_{j_k} : \sum_{k=1}^{M'} l_k = 1, \text{ where } l_k \in [0,1] \text{ for each integer } 1 \leqslant k \leqslant M' \right\}, \tag{5.34}$$

where $\{\mu_{j_k} : k \in \mathbb{N} \cap [1, M']\} = \{\mu_i : i \in \mathbb{N} \cap [1,N]\} \cap \mathcal{E}(T,\phi) \cap \mathcal{P}(X,C) \cap \mathcal{M}^*$. Moreover, by essentially the same proof as Claim 4, we can prove that

$$\mathcal{E}(T,\phi) \cap \mathcal{P}(X,C) = \mathrm{cl}(\mathcal{E}(T,\phi) \cap \mathcal{P}(X,C) \cap \mathcal{M}^*, \mathcal{E}(T,\phi) \cap \mathcal{P}(X, C \smallsetminus \mathrm{Sing}(T))). \tag{5.35}$$

Hence, by (5.35) and (5.34), we apply Corollary 5.9 (i) to the recursively compact set $\mathcal{E}(T,\phi) \cap \mathcal{P}(X, C \smallsetminus \mathrm{Sing}(T))$ and the sequence $\{\mu_{j_k}\}_{k=1}^{M'}$. Then we obtain that $\mathcal{E}(T,\phi) \cap \mathcal{P}(X,C)$ is recursively compact.

Additionally, we assume that $\mathcal{E}(T,\phi) \cap \mathcal{P}(X, C \smallsetminus \mathrm{Sing}(T)) = \{\mu_\phi\}$. Then $\{\mu_\phi\}$ is a recursively compact set. Therefore, it follows from Proposition 3.18 (i) that the measure μ_ϕ is computable. $\square$

6. Applications: Computability of equilibrium states for expanding Thurston maps

In this section, we consider a class of non-uniformly expanding maps on the topological 2-sphere known as *expanding Thurston maps*. On the one hand, there are many available researches on thermodynamic formalism on expanding Thurston maps (see for example, [BM10, BM17, HP09, Li18, Li15, Li17, LS24]). On the other hand, there has been active research on the algorithmic aspects of these maps (see for example, [SY15, RSY20]).

We start with the definition of expanding Thurston maps and go over some key concepts and results. Then in Subsection 6.2, we focus on Misiurewicz–Thurston rational maps and apply Approach I (see Theorem 1.1) to establish Theorem 1.2. Finally, in Subsection 6.3, we study an expanding Thurston map from the barycentric subdivisions and apply Approach II (see Theorem 1.3) to prove Theorem 1.4.

6.1. **Expanding Thurston maps.** In this subsection, we go over some key concepts and results on expanding Thurston maps. For a more thorough treatment of the subject, we refer to [BM17, Li17].

Let S^2 denote an oriented topological 2-sphere and $f\colon S^2 \to S^2$ be a branched covering map. We denote by $\deg_f(x)$ the local degree of f at $x \in S^2$. The *degree* of f is $\deg f = \sum_{x\in f^{-1}(y)} \deg_f(x)$ for $y \in S^2$ and is independent of y.

A point $x \in S^2$ is a *critical point* of f if $\deg_f(x) \geqslant 2$. The set of critical points of f is denoted by $\operatorname{crit} f$. A point $y \in S^2$ is a *postcritical point* of f if $y = f^n(x)$ for some $x \in \operatorname{crit} f$ and $n \in \mathbb{N}$. The set of postcritical points of f is denoted by $\operatorname{post} f$. If the cardinality of $\operatorname{post} f$ is finite, then f is called *postcritically-finite*.

Definition 6.1 (Thurston maps)**.** A Thurston map is a branched covering map $f\colon S^2 \to S^2$ on S^2 with $\deg f \geqslant 2$ and $\operatorname{card}(\operatorname{post} f) < +\infty$.

We can now define expanding Thurston maps.

Definition 6.2 (Expanding Thurston maps)**.** A Thurston map $f\colon S^2 \to S^2$ is called *expanding* if there exists a metric d on S^2 that induces the standard topology on S^2 and a Jordan curve $\mathcal{C} \subseteq S^2$ containing $\operatorname{post} f$ such that

$$\lim_{n\to+\infty} \sup\{\operatorname{diam}_d(X) : X \text{ is a connected component of the set } f^{-n}(S^2 \smallsetminus \mathcal{C})\} = 0.$$

For an expanding Thurston map f, we can fix a particular metric d on S^2 called a *visual metric for f*. Such a metric induces the standard topology on S^2 ([BM17, Proposition 8.3]). For the existence and properties of such a metric, see [BM17, Chapter 8]. For a visual metric d for f, there exists a unique constant $\Lambda > 1$ called the *expansion factor* of d (see [BM17, Chapter 8] for more details).

We summarize the existence and uniqueness of equilibrium states for expanding Thurston maps in the following theorem, which is part of [Li18, Theorem 1.1].

Theorem 6.3 (Li [Li18])**.** *Let $f\colon S^2 \to S^2$ be an expanding Thurston map and d a visual metric on S^2 for f. Let $\phi \in C^{0,\alpha}(S^2, d)$ be a real-valued Hölder continuous function with an exponent $\alpha \in (0, 1]$. Then there exists a unique equilibrium state μ_ϕ for the map f and the potential ϕ.*

The main tool used in [Li18] to develop the thermodynamic formalism for expanding Thurston maps is the Ruelle operator. We recall the definition of the Ruelle operator below and refer the reader to [Li17, Chapter 3.3] for a detailed discussion.

Let $f\colon S^2 \to S^2$ be an expanding Thurston map and $\varphi \in C(S^2)$ be a real-valued continuous function. The *Ruelle operator* $\mathcal{L}_\varphi$ (associated to f and φ) acting on $C(S^2)$ is defined as the following

$$\mathcal{L}_\varphi(u)(x) = \sum_{y\in f^{-1}(x)} \deg_f(y)u(y)\exp(\varphi(y)), \tag{6.1}$$

for each $u \in C(S^2)$. Note that $\mathcal{L}_\varphi$ is a well-defined and continuous operator on $C(S^2)$.

Recall that the measure-theoretic entropy function of a continuous map $T\colon X \to X$ defined on a compact metrizable topological space X is the function $\mu \mapsto h_\mu(T)$ defined on the space $\mathcal{M}(X, T)$.

The following result regarding the upper semi-continuity of the measure-theoretic entropy function for expanding Thurston maps is established in [LS24, Theorem 1.1], extending [Li15, Corollary 1.3].

Theorem 6.4 (Li & Shi [LS24]). *Let $f\colon S^2 \to S^2$ be an expanding Thurston map. Then the measure-theoretic entropy function of f is upper semi-continuous if and only if f has no periodic critical points.*

6.2. **Misiurewicz–Thurston rational maps.** In this subsection, we apply Approach I (see Theorem 1.1) to show the computability of the equilibrium states for Misiurewicz–Thurston rational maps (Theorem 1.2).

A *Misiurewicz–Thurston rational map* is a postcritically-finite rational map on the Riemann sphere $\widehat{\mathbb{C}}$ without periodic critical points. We remark that a postcritically-finite rational map is expanding (in the sense of Definition 6.2) if and only if it has no periodic critical points (see [BM17, Proposition 2.3]).

In the following lemma, we apply some classical results on Newton's Method (see for example, [BCSS98, Section 8]) to design an algorithm computing all the zeros of polynomials. We include the proof for the sake of completeness.

Lemma 6.5. *There exists an algorithm that satisfies the following property:*

For each $m \in \mathbb{N}$, each $n \in \mathbb{N}$, and each complex polynomial p of degree m, this algorithm outputs a sequence $\{q_i\}_{i=1}^m$ of integers satisfying that if $x_1, x_2, \dots, x_m$ are all the zeros of the map p (counting with multiplicity), then there exists a permutation σ on $\{1, 2, \dots, m\}$ such that $\sigma\big(u_{q_{\sigma(i)}}, x_i\big) < 2^{-n}$ for each integer $1 \leqslant i \leqslant m$, where $\{u_j\}_{j\in\mathbb{N}}$ is the effective enumeration of the set $\mathbb{Q}(\widehat{\mathbb{C}})$, after we input the following data in this algorithm:

(i) *an algorithm $\mathcal{A}_p$ computing all the coefficients of the polynomial p,*
(ii) *the integer n.*

Proof. Let $\{s_i\}_{i\in\mathbb{N}}$ be an effective enumeration of the set $\{a+b\mathbf{i} : a, b \in \mathbb{Q}\}$. First, we design an algorithm $M(\cdot,\cdot)$ satisfying the following property. For each polynomial Q, there exists a zero z_0 of Q satisfying that for each $m \in \mathbb{N}$, $M(\mathcal{A}_Q, m)$ outputs a point $l_m \in \mathbb{Q}(\widehat{\mathbb{C}})$ with $\sigma(l_m, z_0) < 2^{-m}$ after we input an algorithm $\mathcal{A}_Q$ computing all the coefficients of the polynomial Q and the integer m.

Begin

(i) Read the integer m and the algorithm $\mathcal{A}_Q$ computing all the coefficients of the polynomial Q
(ii) Set i to be 1.
(iii) **While** $i \geqslant 1$ **do**
   (1) Use the algorithm $\mathcal{A}_Q$ to compute a sequence $\{a_{i,n}\}_{n\in\mathbb{N}}$ of numbers in $\{a + b\mathbf{i} : a, b \in \mathbb{Q}\}$ such that $|a_{i,n} - Q'(s_i)| < 2^{-n}$ for each $n \in \mathbb{N}$.
   (2) Set u to be 1.
   (3) **While** $1 \leqslant u \leqslant i-1$ **do**
      (A) If $|a_{u,i-u}| > 2^{-i+u}$, then use the algorithm $\mathcal{A}_Q$ to compute

$$\gamma(Q, s_u) \coloneqq \sup_{k\geqslant 2} \left|\frac{Q^{(k)}(s_u)}{k!Q'(s_u)}\right|^{\frac{1}{k-1}} \quad \text{and} \quad \beta(Q, s_u) \coloneqq \left|\frac{Q(s_u)}{Q'(s_u)}\right|.$$

      If $\alpha(Q, s_u) \coloneqq \beta(Q, s_u)\gamma(Q, s_u) < \alpha_0$ (here we can select $\alpha_0 = 0.03$, see Remark 6 of [BCSS98, Section 8.2]), then
         (a) Compute an integer k_m satisfying that $k_m > \log_2(m + 4 + \log_2(\beta(Q, s_u)))$.
         (b) Use the algorithm $\mathcal{A}_Q$ to compute and output a point $l_m \in \mathbb{Q}(\widehat{\mathbb{C}})$ with $|l_m - N_Q^{k_m}(s_u)| < 2^{-m-2}$, where $N_Q(z) \coloneqq z - \frac{Q(z)}{Q'(z)}$ for each $z \in \mathbb{C}$.
      (B) Set u to be $u + 1$.
   (4) Set i to be $i + 1$.

End

Fix an integer $m \in \mathbb{N}$, a polynomial Q, and an algorithm $\mathcal{A}_Q$ computing all the coefficients of the polynomial Q. In order to verify the feasibility of the above algorithm, we show that there exists a root $z_0 \in \mathbb{C}$ of Q with $\sigma(l_m, z_0) < 2^{-m}$ for each $m \in \mathbb{N}$. Here l_m is the output of the algorithm $M(\mathcal{A}_Q, m)$.

In Step (iii) (3) (A), we check whether there exists $u \in \mathbb{N}$ such that $|a_{u,i-u}| > 2^{-i+u}$ to check whether $Q'(s_u) \neq 0$. If $Q'(s_u) \neq 0$, then $\gamma(Q, s_u)$ and $\beta(Q, s_u)$ both exist. Since $\{s_i : i \in \mathbb{N}\} = \{a + b\mathbf{i} : a,\, b \in \mathbb{Q}\}$ is dense in $\mathbb{C}$, $\alpha_0 > 0$, and $\beta(Q, \xi) = 0$ for each root $\xi \in \mathbb{C}$ of Q, there exists $u \in \mathbb{N}$ with $\alpha(Q, s_u) < \alpha_0$. If such $u \in \mathbb{N}$ is found in Step (iii) (3) (A), then by Theorem 2 of [BCSS98, Section 8.2], there exists a zero $z_0 \in \mathbb{C}$ of Q satisfying that

$$|N_Q^t(s_u) - z_0| \leqslant \frac{|s_u - z_0|}{2^{2^t - 1}} \leqslant \frac{2\beta(Q, s_u)}{2^{2^t-1}} \quad \text{for each } t \in \mathbb{N}.$$

Combining with the fact that $k_m > \log_2(m + 4 + \log_2(\beta(Q, s_u)))$, this implies that

$$|N_Q^k(s_u) - z_0| \leqslant \frac{2\beta(Q, s_u)}{2^{2^k - 1}} < \frac{2\beta(Q, s_u)}{2^{m+3+\log_2(\beta(Q,s_u))}} = \frac{1}{2^{m+2}}. \tag{6.2}$$

By the definition of the chordal metric σ on $\widehat{\mathbb{C}}$ (see Section 2), it is not hard to see that $\sigma(z, w) \leqslant 2|z - w|$ for each pair of $z,\, w \in \mathbb{C}$. Hence, by $|l_m - N_Q^k(s_u)| < 2^{-m-2}$ and (6.2), we obtain that

$$\sigma(l_m, z_0) \leqslant 2|l_m - z_0| \leqslant 2(|l_m - N_Q^k(s_u)| + |N_Q^k(s_u) - z_0|) < 2^{-m}.$$

Next, we come back to the proof of the original statement. Fix an integer n and a complex polynomial p of degree n. First, we can use the algorithm $M(\mathcal{A}_p, \cdot\,)$ to compute a zero of the polynomial p, say z_0. Then we consider the map $\overline{p}(z) \coloneqq \frac{p(z)}{z - z_0}$. Since $p(z_0) = 0$, $\overline{p}$ is a polynomial of degree $n-1$. Now we claim that we can compute all the coefficients of the polynomial $\overline{p}$ from the point z_0 and all the coefficients of the polynomial p. Indeed, if $p(z) = \sum_{i=0}^n a_i z^i$ and $\overline{p}(z) = \sum_{i=0}^{n-1} b_i z^i$, then it is not hard to see that $b_i = a_{i+1} + z_0 b_{i+1}$ for each integer $0 \leqslant i \leqslant n-1$, where b_n is set to be 0. Hence, we obtain an algorithm $\mathcal{A}_{\overline{p}}$ computing all the coefficients of $\overline{p}$. Then we can use the algorithm $M(\mathcal{A}_{\overline{p}}, \cdot\,)$ to compute a zero of the polynomial $\overline{p}$, i.e., a new zero of the polynomial p. Therefore, we can compute all the zeros of p (counting with multiplicity) recursively. ☐

We now design an algorithm to compute the function $\mathcal{L}_\phi(\mathbb{1})$ (recall (6.1)).

Proposition 6.6. *There exists an algorithm that satisfies the following property:*

For all Misiurewicz–Thurston rational map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, pair of integers n and m, point $x \in \widehat{\mathbb{C}}$, real-valued continuous functions $\phi\colon \widehat{\mathbb{C}} \to \mathbb{R}$ and $u\colon \widehat{\mathbb{C}} \to \mathbb{R}$, this algorithm outputs a rational 2^{-n}-approximation for the value of $\mathcal{L}_\phi^m(u)(x)$, after inputting the following data in this algorithm:

- (i) *an algorithm computing the function ϕ,*
- (ii) *an algorithm computing all the coefficients of the rational map f,*
- (iii) *an oracle of the point x,*
- (iv) *the integers n and m.*

Proof. Since we can compute the function $\mathcal{L}_\phi^m(u)$ by iterating the operator $\mathcal{L}_\phi$ on the function u. Thus it suffices to establish the algorithm in the case of $m = 1$. Let $\{s_i\}_{i \in \mathbb{N}}$ be the effective enumeration of the set $\mathbb{Q}(\widehat{\mathbb{C}})$ and write the rational map g as a quotient of a pair of two polynomials h_1 and h_2 without common roots, i.e., $g(z) = h_1(z)/h_2(z)$ for each $z \in \widehat{\mathbb{C}}$. First, we use data (ii) to find $i \in \mathbb{N}$ such that $s_i \notin f^{-1}(x)$. Then we consider the rational map g given by $g(z) \coloneqq f\bigl(s_i + \frac{1}{z}\bigr)$ for each $z \in \widehat{\mathbb{C}}$. Hence, it follows from some simple computation that $y \in g^{-1}(x)$ if and only if $s_i + \frac{1}{y} \in f^{-1}(x)$, moreover, $\deg_f\bigl(s_i + \frac{1}{y}\bigr) = \deg_g(y)$. Thus, by $s_i \notin f^{-1}(x)$, we obtain that $\infty \notin g^{-1}(x)$.

Now we input the oracle, say φ, of x in the following algorithm $M(\cdot)$ to locate the rough position of x.

Begin

- (i) Set i to be 1.
- (ii) **While** $i \geqslant 1$ **do**
  - (1) If $\sigma\bigl(s_{\varphi(i)}, 0\bigr) \geqslant 2^{-i} + 1$, then outputs 0 and halts.
  - (2) If $\sigma\bigl(s_{\varphi(i)}, \infty\bigr) \geqslant 2^{-i} + 1$, then outputs 1 and halts.
  - (3) Set i to be $i + 1$.

End

We show that this algorithm $M(\varphi)$ will halt eventually. We argue by contradiction and assume that the algorithm $M(\varphi)$ runs forever. Then $\sigma\bigl(s_{\varphi(i)}, 0\bigr) < 2^{-i} + 1$ and $\sigma\bigl(s_{\varphi(i)}, \infty\bigr) < 2^{-i} + 1$ for each $i \in \mathbb{N}$. By $\sigma\bigl(s_{\varphi(i)}, x\bigr) < 2^{-i}$ for each $i \in \mathbb{N}$, it follows from the triangle inequality of the metric σ that $\sigma(x, 0) \leqslant \sigma\bigl(s_{\varphi(i)}, x\bigr) + \sigma\bigl(s_{\varphi(i)}, 0\bigr) < 1 + 2^{1-i}$ for each $i \in \mathbb{N}$. Thus, we obtain that $\sigma(x, 0) \leqslant 1$. Similarly, we have that $\sigma(x, \infty) \leqslant 1$. On the other hand, by the definition of the chordal metric σ (see Section 2), $\overline{B_\sigma}(0, 1) \cap \overline{B_\sigma}(\infty, 1) = \emptyset$. Thus, there exists no point x satisfying that $\sigma(x, 0) \leqslant 1$ and $\sigma(x, \infty) \leqslant 1$. Hence, the algorithm $M(\varphi)$ will halt eventually.

We now prove that the algorithm $M(\cdot)$ satisfies the desired property. Fix an arbitrary oracle φ of the point x. We split the proof into two cases according to the output of the algorithm $M(\varphi)$.

Case 1: The algorithm $M(\varphi)$ outputs 1.

In this case, there exists an integer j with $\sigma\bigl(s_{\varphi(j)}, \infty\bigr) \geqslant 2^{-j} + 1$. Combining the fact that $\sigma\bigl(s_{\varphi(j)}, x\bigr) < 2^{-j}$, this implies that x is not in $B_\sigma(\infty, 1)$. Then $h_1 - xh_2$ is a well-defined polynomial. By $\infty \notin g^{-1}(x)$, we obtain that $y \in g^{-1}(x)$ is equivalent to $h_1(y) - xh_2(y) = 0$. Moreover, it follows from some simple computation that $\deg_g(y) = \deg_{h_1 - xh_2}(y)$ for each $y \in g^{-1}(x)$. Note that by (6.1), we have

$$
\begin{aligned}
\mathcal{L}_\phi(u)(x) &= \sum_{\overline{x} \in f^{-1}(x)} \deg_f(\overline{x}) u(\overline{x}) \exp(\phi(\overline{x})) \\
&= \sum_{y \in g^{-1}(x)} \deg_f(s_i + 1/y) u(s_i + 1/y) \exp(\phi(s_i + 1/y)) \\
&= \sum_{y \in g^{-1}(x)} \deg_g(y) u(s_i + 1/y) \exp(\phi(s_i + 1/y)) \\
&= \sum_{y \in (h_1 - xh_2)^{-1}(0)} \deg_{h_1 - xh_2}(y) u(s_i + 1/y) \exp(\phi(s_i + 1/y)) \\
&= \sum_{y \in (h_1 - xh_2)^{-1}(0)} u(s_i + 1/y) \exp(\phi(s_i + 1/y)),
\end{aligned} \tag{6.3}
$$

where the last expression is counted with multiplicity. Note that $h_1 - xh_2$ is a computable polynomial. Then we can use the algorithms in Lemma 6.5, and [Wei00, Example 4.3.3] to compute a rational 2^{-n}-approximation of the value of $\mathcal{L}_\phi(u)(x)$.

Case 2: The algorithm $M(\varphi)$ outputs 0.

In this case, there exists an integer j with $\sigma\bigl(s_{\varphi(j)}, 0\bigr) \geqslant 2^{-j} + 1$. Combining with the fact that $\sigma\bigl(s_{\varphi(j)}, x\bigr) < 2^{-j}$, this implies that x is not in $B_\sigma(0, 1)$. Then $h_2 - h_1/x$ is a well-defined polynomial. By $\infty \notin g^{-1}(x)$, we obtain that $y \in g^{-1}(x)$ is equivalent to $h_2(y) - h_1(y)/x = 0$. Moreover, it follows from some simple computation that $\deg_g(y) = \deg_{h_2 - h_1/x}(y)$. Similarly, we use the algorithms in Lemma 6.5, and [Wei00, Example 4.3.3] to compute a rational 2^{-n}-approximation of the value of $\mathcal{L}_\phi(u)(x)$.

The proof is complete. □

We now prove the computability of the topological pressure $P(f, \phi)$.

Proposition 6.7. *Let $f \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a computable Misiurewicz–Thurston rational map, d be a visual metric on $\widehat{\mathbb{C}}$ for f, and α be a constant in $(0, 1]$. Then there exists an algorithm with the following property:*

For each $n \in \mathbb{N}$ and each real-valued Hölder continuous function $\phi \in C^{0,\alpha}(\widehat{\mathbb{C}}, d)$ with the exponent α, this algorithm outputs a rational 2^{-n}-approximation for the topological pressure $P(f, \phi)$, after inputting the following data in this algorithm:

(i) *an algorithm computing the potential ϕ,*
(ii) *an algorithm computing all the coefficients of the rational map f,*
(iii) *a rational constant R with $|\phi|_{\alpha,d} \leqslant R$,*

(iv) *the integer n.*

Proof. Denote by $\overline{\phi}$ the function defined by $\overline{\phi}(x) := \phi(x) - P(f,\phi)$ for each $x \in \widehat{\mathbb{C}}$. By [Li18, Lemma 5.15], there exists a constant C (that depends only on f, $\mathcal{C}$, d, and α, not on ϕ) such that

$$\big|\log\big(\mathcal{L}_{\overline{\phi}}^{n}(\mathbb{1})(x)\big)\big| \leqslant C|\phi|_{\alpha,d} \quad \text{for all } x \in \widehat{\mathbb{C}} \text{ and } n \in \mathbb{N}_0. \tag{6.4}$$

We now design an algorithm as follows.

Begin

(1) Compute an integer N with $N > 2^{n+1}CR$.

(2) By Proposition 6.6, we compute and output the value of

$$v_n \approx w_n := N^{-1}\log\big(\mathcal{L}_{\phi}^{N}(\mathbb{1})(x_0)\big)$$

with precision 2^{-n-1}, where $x_0 \in \mathbb{Q}(\widehat{\mathbb{C}})$ is an arbitrary ideal point in $(\widehat{\mathbb{C}},\, \sigma,\, \mathbb{Q}(\widehat{\mathbb{C}}))$.

End

Let us verify that v_n satisfies $|v_n - P(f,\phi)| < 2^{-n}$ for each $n \in \mathbb{N}$. Indeed, by the definitions of w_n and the function $\overline{\phi}$, (6.4), $|\phi|_{\alpha,d} \leqslant R$, and $N > 2^{n+1}CR$, we obtain that

$$\begin{aligned} |w_n - P(f,\phi)| &= \big|N^{-1}\log\big(e^{-NP(f,\phi)}\mathcal{L}_{\phi}^{N}(\mathbb{1})(x_0)\big)\big| < N^{-1}\big|\log\big(\mathcal{L}_{\overline{\phi}}^{N}(\mathbb{1})(x_0)\big)\big| \\ &\leqslant N^{-1}C|\phi|_{\alpha,d} \leqslant N^{-1}CR < 2^{-n-1}. \end{aligned}$$

Combining with the fact that $|v_n - w_n| < 2^{-n-1}$, this implies that $|v_n - P(f,\phi)| < 2^{-n}$. This completes the proof. □

Now we can apply Approach I to prove Theorem 1.2.

Proof of Theorem 1.2. To prove Theorem 1.2, by Theorem 1.1, it suffices to verify that the measure-theoretic entropy function $\nu \mapsto h_\nu(f)$ is upper semi-continuous on $\mathcal{M}(\widehat{\mathbb{C}}, f)$ and the potential $\phi\colon \widehat{\mathbb{C}} \to \mathbb{R}$ satisfy the properties (i) and (ii) in Theorem 1.1.

Indeed, note that by the definition of Misiurewicz–Thurston rational maps, there exists no periodic critical points for the Misiurewicz–Thurston rational map f. Then by Theorem 6.4, we obtain that the measure-theoretic entropy function is upper semi-continuous on $\mathcal{M}(\widehat{\mathbb{C}}, f)$.

Then we apply Proposition 6.7 to show that the topological pressure $P(f,\phi)$ is computable. By [Hei01, Corollary 11.5], there exist two constants $\beta \in (0,1]$ and $C \in \mathbb{Q}^+$ satisfying that

$$\sigma(x,y) \leqslant C d(x,y)^{\beta} \qquad \text{for each pair of } x,\, y \in \widehat{\mathbb{C}}. \tag{6.5}$$

Thus, by $\phi \in C^{0,\alpha}(\widehat{\mathbb{C}},\sigma)$ and $|\phi|_{\alpha,\sigma} \leqslant Q$, we obtain that $\phi \in C^{0,\alpha\beta}(\widehat{\mathbb{C}},d)$, and $|\phi|_{\alpha\beta,d} \leqslant C|\phi|_{\alpha,\sigma} \leqslant CQ$. Hence, by Proposition 6.7, we obtain that $P(f,\phi)$ is computable, i.e., the potential ϕ satisfies the property (ii) in Theorem 1.1.

Denote by $\{s_i\}_{i\in\mathbb{N}}$ the effective enumeration of $\mathbb{Q}(\widehat{\mathbb{C}})$. Then we define for each $i \in \mathbb{N}$ and each $x \in \widehat{\mathbb{C}}$, $f_0(x) := 1$ and $f_i(x) := \sigma(x, s_i)$. Denote by D the set of rational combinations of finite functions in $\{\prod_{j=1}^{m} f_{i_j} : m \in \mathbb{N} \text{ and } i_j \in \mathbb{N}_0 \text{ for each } j \in \mathbb{N}\}$. Thus, by Stone–Weierstrass theorem (see for example, [Fol13, Theorem 4.45]), we obtain that D is dense in $C(\widehat{\mathbb{C}})$. By the existence of computable bijection between $\mathbb{N}^*$ and $\mathbb{N}$, there exists an effective enumeration for D, say $\{\psi_j\}_{j\in\mathbb{N}}$. Then we can compute the following expression for each $j \in \mathbb{N}$:

$$\psi_j = \sum_{k=1}^{l_j} q_{j,k} \prod_{n=1}^{m_{j,k}} f_{i_{j,n,k}}, \text{ where } l_j,\, m_{j,k},\, i_{j,n,k} \in \mathbb{N}_0 \text{ and } q_{j,k} \in \mathbb{Q} \text{ for all } j,\, n,\, k \in \mathbb{N}. \tag{6.6}$$

Claim: The function $i \mapsto P(f,\psi_i)$ is computable on $\mathbb{N}$.

First, we establish a computable function $F\colon \mathbb{N} \to \mathbb{R}$ satisfying that $F(j) \geqslant |\psi_j|_{1,\sigma}$ for each $j \in \mathbb{N}$. By the definition of the function f_i, f_i is Lipschitz continuous function with respect to the chordal metric σ and $|f_i|_{1,\sigma} = 1$ for each $i \in \mathbb{N}$. Moreover, by the definition of the chordal metric σ (see Section 2), we have $\sup_{x,y\in\widehat{\mathbb{C}}}\sigma(x,y)\leqslant 2$. Hence, for each $i\in\mathbb{N}$ and each $x\in\widehat{\mathbb{C}}$, $f_i(x)\leqslant 2$. Combining with the fact that $|f_i|_{1,\sigma}=1$, this implies that for all j, n, $k\in\mathbb{N}$,

$$\begin{aligned}\left|\prod_{n=1}^{m_{j,k}} f_{i_{j,n,k}}(x)-\prod_{n=1}^{m_{j,k}} f_{i_{j,n,k}}(y)\right| &= \sum_{n=1}^{m_{j,k}}\left(\prod_{p=1}^{n-1}\bigl(f_{i_{j,p,k}}(x)\bigr)\cdot\prod_{q=n+1}^{m_{j,k}}\bigl(f_{i_{j,q,k}}(y)\bigr)\cdot\bigl(f_{i_{j,n,k}}(x)-f_{i_{j,n,k}}(y)\bigr)\right)\\ &\leqslant \sum_{n=1}^{m_{j,k}}\left(\prod_{p=1}^{n-1}\bigl|f_{i_{j,p,k}}(x)\bigr|\cdot\prod_{q=n+1}^{m_{j,k}}\bigl|f_{i_{j,q,k}}(y)\bigr|\cdot\bigl|f_{i_{j,n,k}}(x)-f_{i_{j,n,k}}(y)\bigr|\right)\\ &\leqslant 2^{m_{j,k}-1}m_{j,k}\cdot\sigma(x,y).\end{aligned}$$

Thus, by (6.6), we obtain that $|\psi_j|_{1,\sigma}\leqslant\sum_{k=1}^{l_j}\bigl(2^{m_{j,k}-1}m_{j,k}|q_{j,k}|\bigr)$ for each $j\in\mathbb{N}$. Hence, by Definition 3.10, the function $F\colon\mathbb{N}\to\mathbb{R}$ given by $F(j)\coloneqq\sum_{k=1}^{l}\bigl(2^{m_{j,k}-1}m_{j,k}|q_{j,k}|\bigr)$ for each $j\in\mathbb{N}$ is a computable function and satisfies that $F(j)\geqslant|\psi_j|_{1,\sigma}$ for each $j\in\mathbb{N}$.

For each $j\in\mathbb{N}$, since $\psi_j\in C^{0,1}(\widehat{\mathbb{C}},\sigma)$ and $|\psi_j|_{1,\sigma}\leqslant F(j)$, by (6.5), we have $\psi_j\in C^{0,\beta}(X,d)$ and $|\psi_j|_{\beta,d}\leqslant C|\phi_j|_{1,\sigma}\leqslant CF(j)$. Hence, by Proposition 6.7, the function $i\mapsto P(f,\psi_i)$ is computable. Therefore, Claim is now verified.

By Claim, the potential ϕ satisfies property (i) in Theorem 1.1. Recall from Theorem 6.3 that $\mathcal{E}(f,\phi)=\{\mu_\phi\}$. Therefore, by Theorems 1.1, the unique equilibrium state μ_ϕ for the map f and the potential ϕ is a computable measure. ☐

6.3. **An expanding Thurston map from the barycentric subdivisions.** In this subsection, we investigate an expanding Thurston map g derived from the barycentric subdivisions as defined in [BM17, Example 12.21] (denoted there by $\widetilde{f_2}$). Denote by μ the unique measure of maximal entropy for g (recall Theorem 6.3). We show that μ is a computable measure.

Notably, the map g has a fixed critical point. Combining with Theorem 6.4, this implies that the measure-theoretic entropy function of g is not upper semi-continuous. This precludes the possibility of using Approach I (see Theorem 1.1) to prove the computability of the measure μ. Instead, we apply Approach II (see Theorem 1.3) to show that μ is computable.

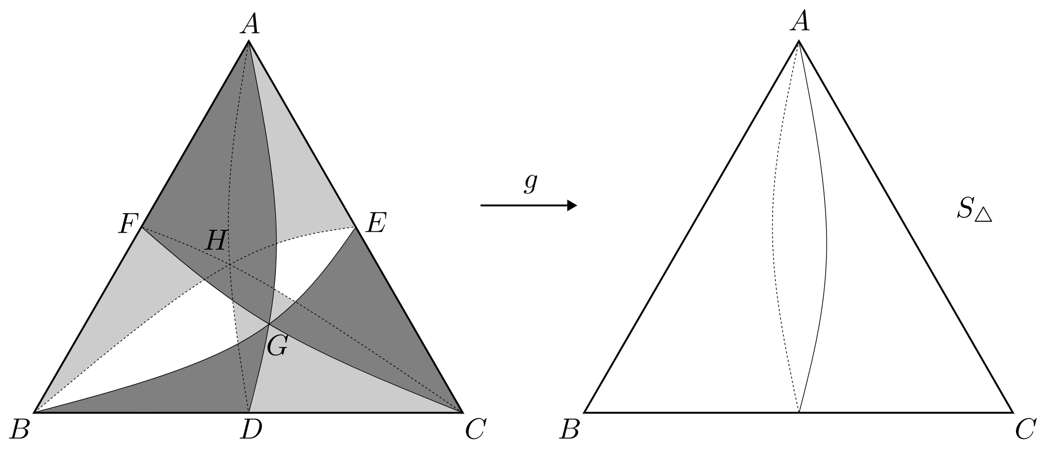

FIGURE 6.1. An expanding Thurston map from the barycentric subdivisions.

For the convenience of the reader, we recall the definition of the map g here. We take two equilateral triangles $\triangle ABC$ and $\triangle A'B'C'$ whose edge lengths are both one, and glue them together along their boundaries with A and A', B and B', and C and C' identified to form a pillow $S_\triangle\coloneqq(\triangle ABC\cup\triangle A'B'C')/\sim$, where the equivalence relation $\sim$ is defined by the aforementioned gluing process. Denote by $\{\triangle_i\}_{i=1}^{12}$ the set of twelve subdivided triangles obtained by the barycentric subdivision of $\triangle ABC$ and $\triangle A'B'C'$ as shown in Figure 6.1 (for example, $\triangle EGC$ is such a subdivided triangle in $\{\triangle_i\}_{i=1}^{12}$).

We now define a piecewise linear map $g\colon S_\triangle\to S_\triangle$ as follows. Let g map $\triangle AGE$ linearly to $\triangle ABC$ on the front side of $S_\triangle$ with $g(A)=A$, $g(E)=C$, and $g(G)=B$. Then we define g on $\triangle EGC$ such that g maps $\triangle EGC$ linearly to $\triangle C'B'A'$ on the back side of $S_\triangle$ with $g(E) = C'$, $g(G) = B'$, and $g(C) = A'$. Inductively, we can obtain a continuous map g on $S_\triangle$ such that g maps each subdivided triangle in $\{\triangle_i\}_{i=1}^{12}$ linearly to the corresponding equilateral triangle in $\{\triangle ABC, \triangle A'B'C'\}$. One can check that $\operatorname{Sing}(g) = \{A, B, C, D, E, F, G, H\}$ and the point A is the only periodic critical point of g. $\operatorname{Sing}(g) \cap \operatorname{Per}(g) = \{A\}$. Moreover, g is an expanding Thurston map (see [BM17, Example 12.21]). Therefore, by Theorem 6.3, there exists a unique measure of maximal entropy μ for the map g.

We next establish a computable structure on $S_\triangle$. For each pair of x, $y \in \triangle ABC$ (resp. $\triangle A'B'C'$), the distance $d_\triangle(x, y)$ between x and y is defined by their Euclidean distance on the triangle. For each $x \in \triangle ABC$ and each $y \in \triangle A'B'C'$, we define

$$d_\triangle(x, y) := \inf_{z\in\mathcal{C}} (d_\triangle(x, z) + d_\triangle(z, y)),$$

where $\mathcal{C} := (\partial\triangle ABC \cup \partial\triangle A'B'C')/\sim$ is a Jordan curve. Moreover, we define

$$\mathbb{Q}(\triangle ABC) := \bigl\{P \in \triangle ABC : \overrightarrow{AP} = \lambda_1\overrightarrow{AB} + \lambda_2\overrightarrow{AC}, \text{ where } \lambda_1,\ \lambda_2 \in \mathbb{Q}_+ \cup \{0\} \text{ satisfy } 0 \leqslant \lambda_1 + \lambda_2 \leqslant 1\bigr\}.$$

Similarly, we can define $\mathbb{Q}(\triangle A'B'C')$. Then by Definition 3.3, $(S_\triangle, d_\triangle, \mathbb{Q}(S_\triangle))$ is a computable metric space, where $\mathbb{Q}(S_\triangle) := (\mathbb{Q}(\triangle ABC) \cup \mathbb{Q}(\triangle A'B'C'))/\sim$.

Proposition 6.8. *In the computable metric space $(S_\triangle, d_\triangle, \mathbb{Q}(S_\triangle))$, the map $g\colon S_\triangle \to S_\triangle$ is computable.*

Proof. We say that a point $P \in S_\triangle$ is represented by $\overrightarrow{\lambda} = \begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} \in \mathbb{R}^{2\times1}$ on $\triangle ABC$ (resp. $\triangle A'B'C'$) if $\overrightarrow{AP} = \lambda_1\overrightarrow{AB} + \lambda_2\overrightarrow{AC}$ (resp. $\overrightarrow{A'P} = \lambda_1\overrightarrow{A'B'} + \lambda_2\overrightarrow{A'C'}$).

Consider an arbitrary integer $1 \leqslant i \leqslant 12$. Since g is affine on $\triangle_i$, there exists a matrix $A_i \in \mathbb{R}^{2\times2}$ and a vertex $\overrightarrow{b_i} \in \mathbb{R}^{2\times1}$ such that $g(x)$ is represented by $A_i\overrightarrow{p} + \overrightarrow{b_i}$ whenever $x \in \triangle_i$ is represented by $\overrightarrow{p} \in \mathbb{R}^{2\times1}$ on the corresponding equilateral triangle in $\{\triangle ABC, \triangle A'B'C'\}$. Note that the matrix A_i and the vertex $\overrightarrow{b_i}$ can be determined based on the three edge points of $\triangle_i$ and all these edge points can be represented by some vertexes in $\mathbb{Q}^{2\times1}$. Then the entries on A_i and $\overrightarrow{b_i}$ are all rational numbers. For example, for the triangle $\triangle AGE$, straightforward computations show that the corresponding matrix and vertex are $\begin{pmatrix}3 & 0\\ -2 & 2\end{pmatrix}$ and $\begin{pmatrix}0\\ 0\end{pmatrix}$, respectively. Moreover, one can check that for each integer $1 \leqslant j \leqslant 12$ and each pair of u, $v \in \triangle_j$,

$$d_\triangle(g(u), g(v)) \leqslant 3d_\triangle(u, v). \tag{6.7}$$

Now we establish an algorithm which, on input an oracle $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Q}(S_\triangle)$ for $x \in S_\triangle$ and an integer m, outputs $y_m \in \mathbb{Q}(S_\triangle)$ such that $d_\triangle(y_m, g(x)) < 2^{-m}$ as follows.

Begin

(i) Read the representation $\overrightarrow{p}$ of x_{m+7}.

(ii) By the definition of $\{\triangle_i\}_{i=1}^{12}$, we can compute $1 \leqslant k \leqslant 12$ such that $x_{m+7} \subseteq \triangle_k$.

(iii) Output $y_m = g(x_{m+7}) = A_k\overrightarrow{p} + \overrightarrow{b_k} \in \mathbb{Q}(S_\triangle)$.

End

Let us verify that $d_\triangle(y_m, g(x)) < 2^{-m}$. Note that $\{\triangle_i\}_{i=1}^{12}$ is a covering of the connected space $S_\triangle$. Then there exists a sequence $\{c_s\}_{s=0}^{l}$ of points in $B_{d_\triangle}\bigl(x_{m+7}, 2^{-m-7}\bigr)$ satisfying the following property:

(i) $c_0 = y_m$ and $c_l = g(x)$.

(ii) For each integer $0 \leqslant s \leqslant l-1$, there exists an integer $i_s \leqslant 12$ such that $\triangle_{i_s}$ contains the points c_s and c_{s+1}.

Hence, by c_s, $c_{s+1} \in \triangle_{i_s}$, it follows from the definition of the map g that $g(c_s)$ and $g(c_{s+1})$ are in the same side of $S_\triangle$. Note that, by $\{c_s\}_{s=0}^{l} \subseteq B_{d_\triangle}\bigl(x_{m+7}, 2^{-m-7}\bigr)$, we have $d_\triangle(c_s, c_{s+1}) < 2\cdot 2^{-m-7} = 2^{-m-6}$.

Then by the triangle inequality of the distance $d_\triangle$, $l \leqslant 12$, and (6.7), we obtain that

$$d_\triangle(y_m, g(x)) \leqslant \sum_{s=0}^{l-1} d_\triangle(g(c_s), g(c_{s+1})) < 3\sum_{s=0}^{l-1} d_\triangle(c_s, c_{s+1}) \leqslant 3l \cdot 2^{-m-6} \leqslant 36 \cdot 2^{-m-6} < 2^{-m}.$$

This completes the proof. □

We now prove the computability of the unique measure of maximal entropy μ for g.

Proof of Theorem 1.4. By Theorem 6.3, there exists a unique equilibrium state for the map g and the potential ϕ_0, where $\phi_0(x) = 0$ for each $x \in S_\triangle$. In particular, μ is the unique measure of maximal entropy.

Now we apply Theorem 1.3 to show that μ is computable. Indeed, by considering more higher levels of barycentric subdivisions, it is not hard to derive from Definition 3.15 that the computable metric space $(S_\triangle,\, d_\triangle,\, \mathbb{Q}(S_\triangle)$ is recursively precompact. Combining with the fact that $(S_\triangle, d_\triangle)$ is complete, by Proposition 3.16, this implies that $S_\triangle$ is recursively compact in $(S_\triangle,\, d_\triangle,\, \mathbb{Q}(S_\triangle))$. Moreover, by the definition of ϕ_0, the function ϕ_0 is computable. By the definition of g and Proposition 6.8, g is a finite-to-one and computable map with $\deg g = 6$.

Then we establish a sequence of uniformly lower semi-computable open sets of X satisfying the properties (a) and (b) in Theorem 1.3 for the recursively compact set $C = S_\triangle$. Now we consider the family of open sets

$$\{\operatorname{int}(\triangle_i \cup \triangle_j) : i,\, j \in \mathbb{N} \cap [1, 12] \text{ such that } \triangle_i \text{ and } \triangle_j \text{ share precisely one common edge}\}.$$

By Definition 3.6, it is not hard to see that this family is consisted by 18 lower semi-computable open sets, say $\{U_j\}_{j=1}^{18}$. Then we obtain that $\bigcup_{j=1}^{18} U_j = S_\triangle \smallsetminus \operatorname{Sing}(g)$. By the definition of map g, for each integer $1 \leqslant j \leqslant 18$, g is injective and open on U_j. Thus, by letting $U_j \coloneqq \emptyset$ for each integer $j > 18$, the sequence $\{U_j\}_{j\in\mathbb{N}}$ satisfies the properties (a) and (b) in Theorem 1.3.

Next, we define $J\colon S_\triangle \to \mathbb{R}$ to be the constant function such that $J(x) = \deg g = 6$ for each $x \in S_\triangle$. Thus, J is computable on $S_\triangle$. By the definition of the map g, for each $x \in g(S_\triangle \smallsetminus \operatorname{Sing}(g)) = S_\triangle \smallsetminus \{A,\, B,\, C\}$, we obtain that $\operatorname{card}(T^{-1}(x) \cap (S_\triangle \smallsetminus \operatorname{Sing}(g))) = 6$ and hence, the function J satisfies property (i) in Theorem 1.3 when $C = S_\triangle$ and $T = g$. Moreover, by [BM17, Corollary 17.2], $h_{\mathrm{top}}(g) = \log(\deg g) = \log 6$. Combining with the fact that $\phi_0(x) = 0$ for each $x \in S_\triangle$, this implies that the function J satisfies property (ii) in Theorem 1.3 when $C = S_\triangle$, $T = g$, and $h(x) = 0$ for each $x \in S_\triangle$.

Finally, we verify that $\mathcal{E}(g, \phi_0) \cap \mathcal{P}(S_\triangle, S_\triangle \smallsetminus \operatorname{Sing}(g)) = \{\mu\}$. To this end, it suffices to show that $\mu \in \mathcal{P}(S_\triangle, S_\triangle \smallsetminus \operatorname{Sing}(g))$, i.e., $\mu(\operatorname{Sing}(g)) = 0$. We argue by contradiction and assume that $\mu(\operatorname{Sing}(g)) > 0$. Combining with the fact that $\operatorname{card}(\operatorname{Sing}(g)) < +\infty$, this implies that $\mu(\{x_0\}) > 0$ for some $x_0 \in \operatorname{Sing}(g)$. Moreover, by [BM17, Theorem 17.1], the measure μ is mixing. Since mixing implies ergodicity, it follows that μ is ergodic. Hence, by [LS24, Lemma 6.3 (iii)], it follows from $\mu(\{x_0\}) > 0$ that x_0 is a periodic point of g and $\mu = \frac{1}{n}\sum_{k=0}^{n-1} \delta_{g^k(x_0)}$, where $n \in \mathbb{N}$ is the period of x_0. Thus, by (3.4), it is not hard to see that $h_\mu(g) = 0$. Indeed, since μ is a measure of maximal entropy, $h_\mu(g) = h_{\mathrm{top}}(g) = \log 6 > 0$. This is a contradiction showing that $\mu(S_\triangle \smallsetminus \operatorname{Sing}(g)) = 1$. Therefore, by Theorem 1.3, μ is a computable measure. □

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